# Dynamical Renormalization Group

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–work in progress–

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10-APR-2012

1. Weak Coupling Limit
2. Feshbach-Schur Map and Level-Shift Operator
3. Approximation at arbitrarily large Time Scales

- Let $\mathfrak{H}$ be a Hilbert space, $H_g = H_0 + gW = H_g^*$ and $P = P^* = P^2$ on $\mathfrak{H}$ such that $[P, H_0] = 0$ and $PWP = 0$.
- Define

$$\widehat{H}_g = H_0 + g^2 K_0 + gP^\perp W P^\perp$$

with $K_0 = PK_0P$ chosen later, so $[P, \widehat{H}_g] = 0$. Then

$$\begin{aligned} & e^{-itH_g} - e^{-it\widehat{H}_g} \\ = \; & ig \int_0^t e^{-i(t-s)\widehat{H}_g} \big( -igK_0 - PWP^\perp - P^\perp WP \big) e^{-isH_g} \, ds, \end{aligned}$$

hence

$$Pe^{-itH_g} - e^{-it\widehat{H}_g} P \;=\; ig \int_0^t e^{-i(t-r)\widehat{H}_g}(-igK_0 - PWP^{\perp})e^{-irH_g}\, dr,$$

$$P^{\perp} e^{-irH_g} P \;=\; -ig \int_0^r e^{-i(r-s)\widehat{H}_g} P^{\perp} W P e^{-isH_g} P\, ds,$$

which yields

$$\begin{aligned} & Pe^{-itH_g} P - e^{-it\widehat{H}_g} P \\ & = \; g^2 \int_0^t ds\, e^{-i(t-s)\widehat{H}_g} \Big[ K_0 - \\ & \quad \Big( \int_0^{t-s} e^{ir\widehat{H}_g} PWP^{\perp} e^{-ir\widehat{H}_g} P^{\perp} WP\, dr \Big) \Big] Pe^{-isH_g} P. \end{aligned}$$

- Assuming the integrability of

$$r \mapsto e^{ir\widehat{H}_g} PWP^\perp e^{-ir\widehat{H}_g} P^\perp WP,$$

we set

$$K(T) := \int_T^\infty e^{ir\widehat{H}_g} PWP^\perp e^{-is\widehat{H}_g} P^\perp WPe^{-ir\widehat{H}_g}\, dr$$

and $K_0 := K(0)$ ($\Rightarrow$ implicit eq. for $K_0$!) and obtain

$$Pe^{-itH_g} P - e^{-it\widehat{H}_g} P \\ = g^2 \int_0^t e^{-i(t-s)\widehat{H}_g} K(t-s)\, Pe^{-isH_g} P\, ds.$$

- We rescale the time, i.e., we set $t := g^{-2}\tau$ and assume $C_1 < \tau < C_2$, for some universal constants $C_1, C_2$. Then

$$\begin{aligned} & Pe^{-ig^{-2}\tau H_g} P - e^{-ig^{-2}\tau \widehat{H}_g} P \\ & \quad = \int_0^\tau e^{-ig^{-2}(\tau-\sigma)\widehat{H}_g} K\Big(\frac{\tau-\sigma}{g^2}\Big) Pe^{-ig^{-2}\sigma H_g} P \, ds. \end{aligned}$$

and thanks to the integrability of $K$, we conclude that

$$Pe^{-ig^{-2}\tau H_g} P \sim e^{-ig^{-2}\tau \widehat{H}_g} P,$$

as $g \to 0$.

- Actually, one takes diagonal part of $K_0$ and defines it with $H_0$.

- WCL goes back to [Van Hove 1951];
- Description above patterned after [Davies 1974];
- The WCL of the evolution operator appears in effective descriptions of quantum dynamics by Boltzmann equation [Erdős 2002, Erdős+Salmholfer+Yau 2004];
- The WCL also appears in effective descriptions of quantum dynamics of
  - free particle coupled to (many) heat baths [De Roeck+Fröhlich 2008],
  - spin-boson model [De Roeck+Kupiainen 2010].

• Simplify assumptions: Let $E$ be an eigenvalue of $H_0$ and $P = \mathbf{1}[H_0 = E]$ the corresponding spectral projection. For $t > 0$,

$$e^{-itH_g} = \frac{-1}{2\pi i} \int_{\mathbb{R}+i\varepsilon} e^{-itz} \frac{dz}{H_g - z},$$

which gives

$$\begin{aligned} Pe^{-itH_g} P &= \frac{-1}{2\pi i} \int_{\mathbb{R}+i\varepsilon} e^{-itz} \, P \, (H_g - z)^{-1} \, P \, dz \\ &= \frac{-1}{2\pi i} \int_{\mathbb{R}+i\varepsilon} e^{-itz} \frac{dz}{F_P(H_g - z)} \, P, \end{aligned}$$

where $F_P$ is the *Feshbach-Schur map* (recall $PWP = 0$),

$$\begin{aligned} & F_P(H_g - z) \\ &= (H_0 - z)P - g^2 PWP^{\perp} \left(H_0 - z + gP^{\perp} WP^{\perp}\right)^{-1} P^{\perp} WP \\ &= \left(E - z - g^2 M(z)\right) P, \end{aligned}$$

with

$$M(z) \ := \ PWP^{\perp}\big(H_0 - z + gP^{\perp}WP^{\perp}\big)^{-1}P^{\perp}WP.$$

being the *Level-Shift operator (LSO)*.

• We expect that $z \approx E + i\varepsilon$ gives the dominant contribution.
So, we compare

$$Pe^{-itH_g}P - e^{-it(E-z-g^2M(z))}\,P$$

$$= \ \frac{-1}{2\pi i}\int_{\mathbb{R}+i\varepsilon} e^{-itz}\Big\{\big(E - z - g^2M(z)\big)^{-1} - \big(E - z - g^2M(E+i\varepsilon)\big)^{-1}\Big\}\,P\,dz$$

$$= \ \frac{-g^2}{2\pi i}\int_{\mathbb{R}+i\varepsilon} e^{-itz}\big(E - z - g^2M(z)\big)^{-1} \big(M(E+i\varepsilon) - M(z)\big)\,\big(E - z - g^2M(E+i\varepsilon)\big)^{-1}\Big\}\,P\,dz.$$

• This idea and the relation between WCL and the LSO has been made precise in [Derezinski+Früboes 2005] and [Derezinski+de Roeck 2007].

• The models to which this has been applied are of Pauli-Fierz type (NR QED):

- For the spin-boson model at positive temperature [Jaksic-Pillet 1996] and [B. Fröhlich+Sigal 2000];
- For atoms at zero temperature [B. Fröhlich+Sigal 1999] and [Hasler+Huber+Herbst 2008];

- We also focus on the Spin-Boson model,

$$H_g = H_0 + gW, \quad H_0 = H_{\mathrm{at}} \otimes \mathbf{1} + \mathbf{1} \otimes H_{\mathrm{f}}$$

$$H_{\mathrm{at}} = \begin{pmatrix} E_1 & 0 \\ 0 & E_0 \end{pmatrix}, \quad H_{\mathrm{f}} = \int |k| \, a_k^* a_k \, d^3 k,$$

$$W = \int \Big\{ G(k) \otimes a_k^* + G^*(k) \otimes a_k \Big\} \, d^3 k,$$

$$G(k) = \begin{pmatrix} 0 & b_0(k) \\ b_1(k) & 0 \end{pmatrix},$$

$$P = \mathbf{1} \otimes P_\Omega.$$

• We assume that $\theta \mapsto H_g(\theta)$ is dilation analytic. Recall that $H_g(\theta) = U(\theta) H_g U^{-1}(\theta)$, where

$$\begin{aligned} U(\theta)(A_{\mathrm{at}} \otimes a_k^*) U^{-1}(\theta) &= A_{\mathrm{at}} \otimes e^{3\theta/2} a_{e^\theta k}^*, \\ U(\theta)(\varphi_{\mathrm{at}} \otimes \Omega) &= \varphi_{\mathrm{at}} \otimes \Omega. \end{aligned}$$

• Note that $\theta \mapsto P e^{-itH_g(\theta)} P \in \mathbb{C}^{2\times 2}$ is analytic on $\{|\theta| < \theta_0\}$ and $P e^{-itH_g(\theta)} P = P e^{-itH_g(0} P$, for $\theta \in \mathbb{R}$. Hence,

$$\forall\, |\theta| < \theta_0 : \quad P e^{-itH_g(\theta)} P \;=\; P e^{-itH_g(0} P.$$

Thm. 1: Assume $G(k) \equiv G(|k|)$ to be dilation analytic and to obey $\|G(k)\|_{\mathcal{B}(\mathfrak{H}_{\rm at})} \leq |k|^{-1/2}$. Then, for all $0 < \mu < 2$, $0 < \nu < \min\{1, \frac{1}{\mu} - \frac{1}{2}\}$:

$$\Big\| \mathbf{1}_{H_{\rm f}<\rho} e^{-itH_g(\theta)} \mathbf{1}_{H_{\rm f}<\rho} \; - \; e^{-itK_g(H_{\rm f},\theta)} \otimes \mathbf{1}_{H_{\rm f}<\rho} \Big\| \\ \leq \; C_1 \Big[ e^{-c_2 t} + e^{C_1 g^2 \rho^\nu t} \big( \rho^{\frac{1-\nu}{2}} + g^{1/2} \rho^{-\frac{1}{2}(\nu+\frac{1}{2})} \big) \Big],$$

where $\rho = g^\mu$ and $K_g(H_{\rm f},\theta)_{j\ell} = \delta_{j\ell} K_g^{(\ell)(H_{\rm f},\theta)}$.

Cor. 2: Choose $\mu := \frac{2}{3}$, $\nu := \frac{1}{2}$. Then

$$\left\| P\, e^{-ig^{-2}\tau H_g}\, P \;-\; e^{-i\tau K_g} \otimes P_\Omega \right\| \leq \; C_1 \Big[ e^{-c_2 g^{-2}\tau} \;+\; g^{\frac{1}{6}}\, e^{g^{1/3}\tau} \big( \rho^{\frac{1-\nu}{2}} + g^{1/2} \rho^{-\frac{1}{2}(\nu+\frac{1}{2})} \big) \Big],$$

where $(K_g)_{j\ell} = \delta_{j\ell} K_g^{(\ell)}$ $K_g^{(\ell)} = g^{-2} E_\ell - \Lambda_\ell + \mathcal{O}(g^{2/3})$, and

$$\Lambda_\ell \;=\; \int_0^\infty \frac{k^2\, |b_\ell(k)|^2\, dk}{E_{1-\ell} - E_\ell + |k| - i0}.$$

- Thm. 1 and Cor. 2 reproduce Davies' result and give quantitative bounds.
- Conceptual Difficulties:
  - Formulation of quantitative bounds in general?
  - Generalization of WCL to time scales $\sim g^{-n}\tau$, for arbitrary $n \in \mathbb{N}$? (various proposals, e.g., Kossakowski)

• Study time evolution of effective Hamiltonian ($\theta = i\vartheta$, $\vartheta > 0$) on $\mathfrak{H}_{red} := \mathbf{1}_{H_f < \rho} \mathcal{F}$,

$$
\begin{aligned}
H &= e^{-\theta} H_f + W, \\
W &= \sum_{m+n \geq 1} \mathbf{1}_{H_f < \rho} W_{m,n} \mathbf{1}_{H_f < \rho} \\
W_{m,n} &= \int_{B_1^{m+n}} a^*(k^{(m)}) \, w_{m,n}\big[H_f; k^{(m)}; \tilde{k}^{(n)}\big] \, a(\tilde{k}^{(n)}) \, d^m k \, d^n \tilde{k}, \\
a(\tilde{k}^{(n)}) &= a_{\tilde{k}_1} a_{\tilde{k}_1} \cdots a_{\tilde{k}_n},
\end{aligned}
$$

with $\underline{w} = (w_{m,n})_{m+n \geq 1}$ obeying

$$
\|\underline{w}\|_{\mu,\xi}^{\#} \leq g \leq \delta_0,
$$

as defined in [BCFS] = [B.+Chen+Fröhlich+Sigal 2003].

• In [BCFS] it was proved that the isospectral RG-map $\mathcal{R}_\rho$ based on the Feshbach-Schur map is applicable and yields a sequence of effective Hamiltonians:

$$H - z \;=:\; H_{(0)}[z] \;=\; T_{(0)}[H_{\mathrm{f}}; z] + W_{(0)}[z] - E_{(0)}[z],$$

where $T_{(0)}[H_{\mathrm{f}}; z] := e^{-\theta} H_{\mathrm{f}}$, $W_{(0)}[z] := W$, $E_{(0)}[z] := z$, $|z| < \frac{1}{2}$, and

$$H_{(n+1)}[z] \;:=\; \mathcal{R}_\rho\big(H_{(n+1)}\big)[z].$$

Then [BCFS]

$$H_{(n)}[z] \;=\; T_{(n)}[H_{\mathrm{f}}; z] + W_{(n)}[z] - E_{(n)}[z],$$

with

$$\big\| T_{(n)}[r; z] - e^{-\theta} r \big\| \;\leq\; 2g,$$

$$|E_{(n)}[z] - z| \;\leq\; \rho^{\mu n} g, \quad \|\underline{w}_{(n)}\|^{\#}_{\mu,\xi} \;\leq\; \rho^{\mu n} g,$$

for all $|z| \leq \frac{1}{2}$.

Thm. 3: If $\tau \gg n|\ln(\rho)|\vartheta^{-1}$ then

$$\Big\| \mathbf{1}_{H_f<\rho^n}\, e^{-i\rho^{-n}\tau H}\, \mathbf{1}_{H_f<\rho^n} \;-\; \frac{\rho^{-n}}{M_{(n)}(\rho^{-n}H_f)}\, e^{-i\rho^{-n}\tau Z_{(n)}(\rho^{-n}H_f)}\, \mathbf{1}_{H_f<\rho^n} \Big\|$$

$$\leq \; C_1\, e^{-\mu_{(n)}\rho^{-n}\tau} \big( e^{-c_2\tau} \,+\, g\, \rho^{\mu n-1}\, e^{C_1\rho\tau} \big),$$

for some $0 < c_2 \leq C_1 < \infty$, where $M_{(n)}$ and $Z_{(n)}$ are determined from $T_{(n)}$ and $E_{(n)}$ by a contour integral and

$$\mu_{(n)} \;:=\; -\operatorname{Im}\Big\{ E_{(1)}^{-1}\Big(\rho E_{(2)}^{-1}\big(\cdots \rho E_{(n-1)}^{-1}(0)\cdots\big)\Big)\Big\}$$

is the imaginary part of the $n^{th}$-order approximation to the (resonance) eigenvalue of $H$.

Work in progress and open problems:

- Generalization from 1- to $N$-level atom, yielding $H_{(0)}$ after elimination of field energies $> 1$,
- Estimates directly on the propagator without using Laplace transform.