# Resonant Delocalization for Operators with Random Potential on Tree Graphs

1. *Extended States in a Lifshitz Tail Regime*
2. *Absence of Mobility Edge for Bounded Potentials at Weak Disorder*
3. *Ballistic Evolution throughout ac spectrum (on tree graphs)*

Michael Aizenman — Princeton Univ.

Simone Warzel — TU - Munich

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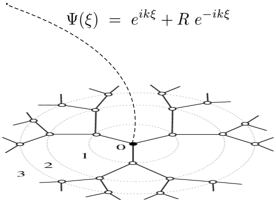
# Coherent Transport in the presence of Disorder

Consider the random Schrödinger operator

$$\boxed{H_\lambda(\omega) := T + \lambda\, V(\omega)}$$

on the $\ell^2$ space of a graph, and the unitary evolution it generates: $U(t) := e^{-itH_\lambda(\omega)}$.

Our focus here will be on the case of homogeneous tree graphs ($\mathbb{T}$), whose degree is denoted $K+1$.




$T$ - adjacency operator: $(T\psi)(x) := \sum_{\text{dist}(x,y)=1} \psi(y)$

$V(\omega)$ - random potential: $V(x;\cdot)$, $x \in \mathbb{T}$, i.i.d. random variables,

$\mathbb{P}(V(0) \in dv) = \varrho(v)\, dv$, with $\varrho$ bounded, piecewise continuous and monotone

*In this talk, special emphasis on two cases:*

1. supp $\varrho = \mathbb{R}$ (e.g. Gaussian or Cauchy)
2. supp $\varrho = [-1, 1]$ (e.g. equidistributed)

$\lambda$ - the disorder parameter

$T$ - absolutely continuous spectrum

extended (generalized) eigenfunctions ($\Psi_E \notin \ell^2(\mathbb{B})$)

ballistic evolution: $\langle |x(t)| \rangle \approx c\,t$

$V(\omega)$ - pure-point spectrum, $\sigma(V(\omega)) = \{V_x(\omega)\}_{x \in \mathbb{B}}$

localized eigenstates ( $\{\delta_x\}_{x \in \mathbb{B}}$)

dynamical localization: $\langle |x(t)| \rangle = \text{Const.}$

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Question: - what are the spectral and dynamical properties of

$$\boxed{H(\omega) = T + \lambda V(\omega)} \quad ?$$

# Significance of the *absolutely cont. spectrum*

The condition $\boxed{\operatorname{Im} G(0,0;E+i0,\omega)\ >\ 0}$ $(*)$

is relevant from both the spectral and dynamical perspectives:

- $(*) \Longrightarrow$ current can be conducted through the graph to infinity ([MD]):

$$\boxed{|R(k,\omega)| < 1 \iff \operatorname{Im} G(0,0;E+i0,\omega)\ >\ 0}$$

- $\pi^{-1} \operatorname{Im} G(u,u;E+i0,\omega)$ is the density of the ac component of the spectral measure $\mu_x(dE)$ associated with the state $\Psi_u(x) = \delta_{x,u}$:

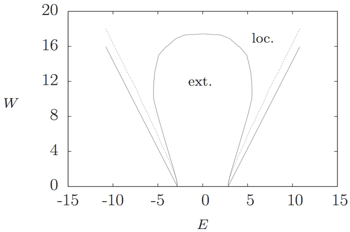
$$\langle x|\, F(H)\, |x\rangle \ = \int_{\mathbb{R}} F(E)\, \mu_x(dE)$$

$$\mu_x(dE) \ = \ \mu_x^{pp}(dE) + \mu_x^{ac}(dE) + \mu_x^{sc}(dE)$$

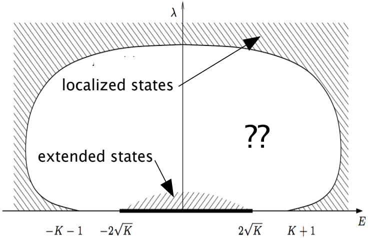
with $\mu_x^{pp}(dE) = \sum_n |\Psi_n(x)|^2\, \delta_{E_n}$,

and $\boxed{\mu_x^{ac}(dE) = \frac{1}{\pi} \operatorname{Im} G(x,x;E+i0,\omega)\, dE}$

# The Expected Mobility Edge




expected mobility edge
for a bounded potential




phase diagram for
unbounded potential

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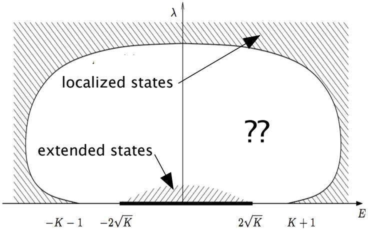
- Among the earliest studied models of the Anderson localization And. '58
  Abou-Chacra/Anderson/Thouless '73, Abou-Chacra/Thouless '74
- Motivation: Relatively more accessible compared to $\mathbb{Z}^d$.
  Self-consistent approach to localization becomes exact ($\neq$ solvable !).
- Renewed interest due to analogies which were drawn with configuration spaces of systems of many particles Altshuler/Gefen/Kamenev/Levitov '97 , (cf. Pal/Huse'11)
- Numerical work: Miller/Derrida '94, Biroli/Semerjian/Tarzia '10




$$\boxed{H \;=\; T \;+\; \lambda V}$$

supp $\varrho = \mathbb{R}$

$\int_{\mathbb{R}} v\, \varrho(v)\, dv \;=0$

$+$ reg. cond.

- Spectrum of the Laplacian on $\ell^2(\mathbb{B})$: $\sigma(T) \;=\; \big[-2\sqrt{K}, 2\sqrt{K}\big]$

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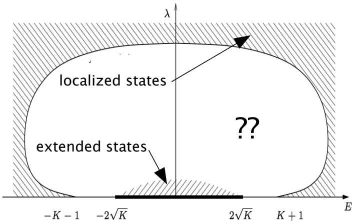
1. Ergodicity $\Rightarrow$ $\boxed{\sigma(H_\lambda(\omega)) \overset{a.s.}{=} \sigma(T) \;+\; \lambda \text{ supp } \rho}$ — Kunz/Souillard '78
2. pure-point spectrum at strong disorder, and at large energies — Aizenman/Molchanov '93, Aiz. '94
3. abs. cont. spectrum for weak disorder at energies within $\sigma(T)$ — Klein '94, Aiz./Sims/Warzel '05, Froese/Hasler/Spitzer '06

# The long open puzzle




supp $\varrho = \mathbb{R}$

Question: Where is the edge of the localization regime, in particular, at weak disorder?

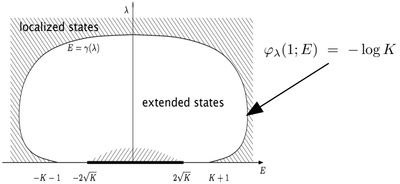
Note: for energies $E$ outside $\sigma(T) = [-2\sqrt{K}, 2\sqrt{K}]$, the mDOS ($\rho_{\mathrm{DOS}}$) vanishes at weak disorder to all orders in perturbation theory (Lifshitz tail regime)

E.g., in case of the Gaussian random potential:

$\rho_{\mathrm{DOS}}(E) \approx \exp\left(-C(E)/\lambda^2\right)$ as $\lambda \downarrow 0$ for $E \notin \sigma(T)$.

# A somewhat surprising answer

**Theorem:** In the case of unbounded random potential ($\operatorname{supp}\rho = \mathbb{R}$, etc.), for $\lambda > 0$ the ac spectrum immediately extends up to $E = \pm(K+1)$, in particular, into the regime of Lifshitz tails.




More can be said in terms of the Green function

$$G(0,x;E) := \left\langle \delta_0, (H - E - i0)^{-1}\delta_x \right\rangle$$

and its moment generating function

$$\text{for } s < 1: \qquad \varphi_\lambda(s;E) \quad := \quad \lim_{|x|\to\infty} \frac{\log \mathbb{E}\left[|G_\lambda(0,x;E+i0)|^s\right]}{|x|}$$

$$\text{for } s = 1: \qquad \varphi_\lambda(1;E) \quad := \quad \lim_{s\nearrow 1} \varphi_\lambda(s;E)$$

# (Almost) complementary criteria for pp and ac spectra

Assumptions: $\varrho(V) > 0$ on $\mathbb{R}$, $\int |v|^\tau \varrho(v) dv < \infty$ for some $\tau > 0$, etc.

Theorem 1 (localization [Aiz./Molchanov '93, Aiz. '94])

- *If for all (or a.e.) energies $E$ in some interval $I \subset \mathbb{R}$*

$$\boxed{\varphi_\lambda(1; E) < -\log K} \qquad (1)$$

*then $H(\omega)$ has only pure point (localized) spectrum in that interval.*

- *Furthermore, at weak disorder (1) holds for energies $|E| > (K+1)$.*

The new, *complementary*, statement:

Theorem 2 (delocalization [Aiz./Warzel '10, '11])

- Under the above assumptions on $\rho$, at energies at which

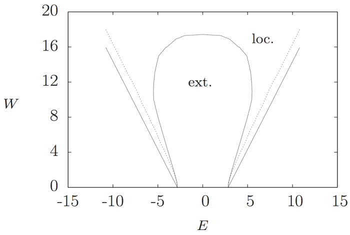
$$\boxed{\varphi_\lambda(1; E) > -\log K} \qquad (2)$$

one has: $\boxed{\operatorname{Im} G(x, x; E + i0) \; > \; 0\,;}$

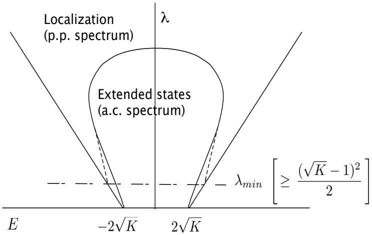
- ($\Longrightarrow$) if (2) holds for a positive measure of energies $E \in I$, then $H(\omega)$ has absolutely continuous (delocalized) spectrum in that interval.

Added by M. Shamis: One may conclude from Thm. 2 that if (2) holds for almost every $E \in I$ then then $H(\omega)$ has only ac spectrum in $I$.

# The 2nd surprise: absence of (the expected) mobility edge at weak disorder




prev. published numerical results




sketch of the corrected phase diagram

## Theorem 3

For $H_\lambda$ as in Theorem 2, with a bounded potential ($\|V\|_\infty = 1$) of regular distribution, for any

$$\lambda < (\sqrt{K} - 1)^2/2$$

there are intervals of with $\delta(\lambda) > 0$ reaching the edges of the spectrum throughout which the random operator has (*a.s*) only purely absolutely continuous spectrum.

(go to refs)

# Criteria for continuous spectrum

I. *For* $H(\omega) = H_0 + \lambda V(\omega)$, *with random potential as above:*

*if* $\forall E \in I$, *almost surely:*

$$\sum_y |G(x, y; E + i0)|^2 = \infty$$

*then within this interval* $H(\omega)$ *has (a.s.) only continuous spectrum* (Simon/Wolff '86).

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II. *For* $H(\omega)$, *as above, if for some interval* $I \subset \mathbb{R}$*:*

$$\operatorname{Im} G(0, 0; E + i0), \omega) > 0$$

*for* $\mu(d\omega) \times dE$ *almost every* $(\omega, E)$*,*

*then within this interval* $H(\omega)$ *has (a.s.) only purely 'ac' spectrum*
*(Aronszajn, cf. Jaksic/Last '86)*

# Essential tools for analysis on trees (and perhaps beyond)

1. "Current conservation" ($J_{u,v} = \ldots$) $\Longrightarrow$

$$\operatorname{Im} G(0,0;E+i0) \;=\; \sum_{x:|x|=n} |G(0,x_-;E+i0)|^2 \; \operatorname{Im} G(x,x;E+i0)$$

2. On tree graphs the Green function factorizes:

$$G(0,x;\zeta) \;=\; \prod_{0 \preceq u \preceq x} \Gamma(u;\zeta), \qquad \zeta \equiv E + i\eta \in \mathbb{C}^+ \,.$$

where $\Gamma(u;\zeta) := \left\langle \delta_u, (H_{\mathbb{T}_u^+} - \zeta)^{-1} \delta_u \right\rangle$ is the resolvent of the operator restricted to the tree forward to $u$. (LERW analogy!!)

$\Longrightarrow$ the typical behavior is: $|G(0,x;\zeta)| \;\approx\; e^{-L_\lambda(\zeta)\mathrm{dist}(0,x)}$

with $\boxed{L_\lambda(\zeta) := -\mathbb{E}(\log|\Gamma(u;\zeta)|)}$ (the 'Lyapunov exponent').

3. A recursion relation holds:

$$\Gamma(x;\zeta) = \left( \lambda\, V(x) - \zeta - \sum_{y \in \mathcal{N}(x)} \Gamma(y;\zeta) \right)^{-1}$$

# Heuristics [perhaps the main message in this talk (!)]

The mechanism at work here: fluctuation enabled resonant tunneling:

States which locally appear to be localized have arbitrarily close energy gaps ($\Delta E$)

with other states (at distances $R$), to which the tunneling amplitudes are

exponentially small (as $\approx e^{-L_\lambda(E)R}$) .

Mixing between two levels occurs if $\boxed{\Delta E \ll e^{-L(\lambda(E)R}}$.

Since the volume grows exponentially fast (as $K^R$),

extended states will form in spectral regimes with $\boxed{L_\lambda(E) < \log K}$

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Essential enabling conditions:

- local fluctuations in the self energy
- the exponential growth of the configuration space volume

For the tight criteria use is also made of the large deviations theory (applied to the Green function).

# Proof by contradiction, based on Resonances

We say that $x$ resonates with 0 at energy $E$, and inverse resonance length $\gamma$, if

1. $\left|\frac{G(0,x;E+i0)}{G(x,x;E+i0)}\right| \geq e^{-\gamma\,\mathrm{dist}(0,x)}$
2. $|G(x,x;E+i0)| = |\lambda V(x) - \sigma(x;E)|^{-1} \geq e^{\gamma\,\mathrm{dist}(0,x)}$,

   with $\sigma(x;E)$ the self-energy.

**Key observations**

- The event $\{1\}$ does not depend on $V(x)$, and neither does $\sigma(x;E)$.
- Under the assumption of no ac spectrum: $\mathbb{P}(\{2\}) \approx e^{-\gamma \mathrm{dist}(0,x)}$.

Hence: if $e^{-\gamma R}\,|\mathcal{S}_R|\,\mathbb{P}\left(\left|\frac{G(0,x;E+i0)}{G(x,x;E+i0)}\right| \geq e^{-\gamma R}\right) \to \infty$,

then the number of resonant sites $N_{R,\gamma}$ on $\mathcal{S}_R := \{\mathrm{dist}(x,0)=R\}$

satisfies $\boxed{\mathbb{E}\left[N_{R,\gamma}\right] \to \infty}$

For the proof of cont. spectrum one needs more: $\boxed{\mathbb{P}\left(N_{R,\gamma} \geq 1\right) \geq p_0 > 0}$

(uniformly in $R$). For this, we employ the $2^{nd}$-moment test: $\mathbb{P}(N \geq 1) \geq \frac{\mathbb{E}[N]^2}{\mathbb{E}[N^2]}$

# Also used: continuity properties of the Lyapunov exponent

A significant observation: at $\lambda = 0$ the condition $\boxed{L_0(E) < \log K}$ holds wherever $|E| < (K+1)$  (i.e. within a larger set than $\sigma(H_0)$).

- Weak continuity: $\lim_{\lambda \downarrow 0} \int_I L_\lambda(E)\, dE = \int_I L_0(E)\, dE$

  implies *ac* spectrum in $(-(K+1), K+1)$ for $\lambda$ small in case $\operatorname{supp} \varrho = \mathbb{R}$.

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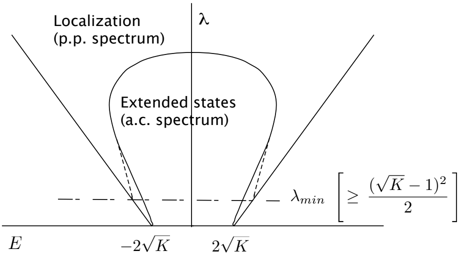
- For bounded potentials , $\operatorname{supp} \varrho = [-\frac{1}{2}, \frac{1}{2}]$ we show:

$$\limsup_{E \downarrow E_\lambda} L_\lambda(E) \le L_0(E_\lambda - \lambda) \qquad \text{(with } E_\lambda := \inf \Sigma_\lambda\text{)}.$$




This implies pure *ac* spectrum near the spectral edges for small $\lambda$!

# Suggested picture, and implications for the time evolution

• Note: It appears that at short distances (& times) there is a qualitative difference in the nature of the extended states (and presumably also time evolution) between this regime, and the perturbative regime which was successfully studied earlier.

• We do not however expect a second sharply defined transition line, but rather a gradual crossover characterized by a "tunneling distance".

• It is also proven that throughout the *ac* regime the time evolution is ballistic (previously this was established by A. Klein '98 for the "perturbative regime").

# Bounds on the time evolution

Notation:

$$P_{\psi,t}(x) := \left| \left( e^{-itH} \psi \right) (x) \right|^2 .$$

A known general upper bound on the speed of propagation (baby version of the Lieb - Robinson bound):

$$\Pr_{\delta_0,t}(d(x,0) > vt) \; := \sum_{x:\, d(x,0)>vt} P_{\delta_0,t}(x) \; \leq \; e^{-\mu t(v-\hat{v})}$$

at some $\mu > 0$, with a finite speed $\hat{v} < \infty$ which does not depend on the potential $V$. This implies in particular the ballistic upper bounds (for all $0 < p < \infty$):

$$\boxed{\mathbb{E}\left[|x(t)|^p\right] \; \leq \; C_p \, t^p \, .}$$

Spectral analysis (Wiener, RAGE, Guarneri, ...) yields bound on the time-averaged transition probability, over time $T \equiv (2\eta)^-1$, based on:

$$\begin{aligned} \widehat{P}_{\psi,T}(x) \quad &:= \quad \int_0^\infty e^{-t/T} \, P_{\psi,t}(x) \, \frac{dt}{T} \\ &= \quad \frac{\eta}{\pi} \int \left| \left( (H - E - i\eta)^{-1} \psi \right) (x) \right|^2 \, dE \end{aligned}$$

# Ballistic evolution throughout the abs. cont. spectrum

Theorem: On a regular tree graph, for any initial state $\psi = f(H)\delta_0$ with $f \in L^2(\mathbb{R})$, supp $f \subset \sigma_{ac}(H)$:

for all $b > 0$:

$$\mathbb{E}\left[\widehat{\Pr}_{\psi,T}(|x| < b\,T)\right] \;\leq\; C(f)\,b \;+\; o(1/T)\,.$$

with some $C(f) < \infty$.

In particular: all moments obey also ballistic lower bounds:

$$\boxed{\widehat{\mathbb{E}}_T\left[|x|^p\right] \;\geq\; \widehat{C}_p\,T^p}$$

It should however be noted that the above does not really contradict the expected rule that in the presence of disorder *absolutely continuous* spectrum yields diffusive behavior – classical diffusion on a tree is also ballistic (!)

References: the results presented here were derived in the joint works with S. Warzel:

1. “Extended States in a Lifshitz Tail Regime for Random Schrödinger Operators on Trees", Phys. Rev. Lett. **106**, 136804 (2011).

2. “Absence of Mobility Edge for the Anderson Random Potential on Tree Graphs at Weak Disorder”, Euro. Phys. Lett. **96**, 37004 (2011).

3. “Resonant delocalization for random Schrödinger operators on tree graphs", JEMS (to appear)

4. “Absolutely continuous spectrum implies ballistic transport for quantum particles in a random potential on tree graphs”, J. Math. Phys. (to appear)

(all available on arXiv)

(the corrected phase diagram)