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PDE II
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Excercise Sheet 5 for 12. 06. 2017 (6 exercises for 2 weeks!)

Let $d \in \mathbb{N}$.

5.1. Let A be a measurable subset of $\mathbb{R}^d$ with finite Lebesgue measure. Prove that for every sequence $(f_n)_{n\in\mathbb{N}}$ converging weakly to f in $H^1(\mathbb{R}^d)$ the sequence $(1_A f_n)_{n\in\mathbb{N}}$ converges to $1_A f$ strongly in $L^p(\mathbb{R}^d)$ for all $p \in [2, p_{\max})$,

$$p_{\max} := \begin{cases} 2d/(d-2), & \text{if } d \geqslant 3; \\ \infty, & \text{if } d \in \{1,2\}. \end{cases}$$

5.2. Prove that $H^2(\mathbb{R}^3)$ boundedly embeds into $C(\mathbb{R}^3)$. Note that the statements about the embeddings of $W^{k,p}(\mathbb{R}^d)$ were only proved in the class for $k = 1$ and thus cannot be used in the solution for $k > 1$.

5.3. Let $d \in \{1, 2\}$ and $V \in L^1(\mathbb{R}^d)$ be real-valued with $\int_{\mathbb{R}^d} V(x)\,\mathrm{d}x < 0$. Prove that

$$E_V := \inf \left\{ \int_{\mathbb{R}^d} \Big(\big|\nabla\varphi(x)\big|^2 + V(x)\big|\varphi(x)\big|^2 \Big) \mathrm{d}x : \varphi \in H^1(\mathbb{R}^d), \|\varphi\|_{L^2} = 1 \right\} < 0.$$

5.4. Prove that there exists $C_d > 0$ such that every radial $f \in H^1(\mathbb{R}^d)$ (i.e. $f(x) = f\big(|x|\big)$ for a.e. $x \in \mathbb{R}^d$) satisfies

$$\big|f(x)\big| \leqslant C_d |x|^{(1-d)/2} \|f\|_{H^1} \quad \text{for a.e. } x \in \mathbb{R}^d \text{ with } |x| > 1.$$

5.5. Prove that if a sequence $(f_n)_{n\in\mathbb{N}}$ converges weakly to f in $H^1(\mathbb{R}^d)$, then $\big(|f_n|\big)_{n\in\mathbb{N}}$ converges weakly to $|f|$ in $H^1(\mathbb{R}^d)$.

5.6. Find a constant $a > 0$ such that $f(x) := |x|^{-a}$ solves the equation

$$\Big(-\Delta - \frac{1}{4|x|^2} \Big) f(x) = 0, \quad \text{for all } x \in \mathbb{R}^3 \setminus \{0\}.$$

Use the Perron-Frobenius principle to conclude the Hardy inequality

$$\int_{\mathbb{R}^3} \big|(\nabla u)(x)\big|^2 \,\mathrm{d}x \geqslant \frac{1}{4} \int_{\mathbb{R}^3} \frac{\big|u(x)\big|^2}{|x|^2} \,\mathrm{d}x, \quad \text{for all } u \in H^1(\mathbb{R}^3).$$