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PDE II
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Excercise Sheet 1 for 8. 05. 2017

Let (Ω, Σ, μ) be a measure space.

1.1. Prove the following extension of Hölder's inequality:
For $m \in \mathbb{N}$ let $f_j \in L^{p_j}(\Omega)$, $j = 1, \ldots, m$ with $\sum_{j=1}^m 1/p_j = 1$. Then

$$\left| \int_\Omega \prod_{j=1}^m f_j \mathrm{d}\mu \right| \leqslant \prod_{j=1}^m \|f_j\|_{p_j}.$$

1.2. Prove the following extension of Young's inequality:
Let $p, q, r \in [1, \infty]$ with $1/p + 1/q = 1 + 1/r$. Then for any $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$ the convolution $f * g$ belongs to $L^r(\mathbb{R}^d)$ and $\|f * g\|_r \leqslant \|f\|_p \|g\|_q$.

1.3. For $p \in [1, \infty)$ let $(f_j)_{j \in \mathbb{N}}$ be a sequence in $L^p(\Omega)$ such that there exists $f \in L^p(\Omega)$ with $\lim_{j \to \infty} f_j(x) = f(x)$ for μ-a.e. $x \in \Omega$ and $\lim_{j \to \infty} \|f_j\|_p = \|f\|_p$. Prove that $(f_j)_{j \in \mathbb{N}}$ converges to f strongly in $L^p(\Omega)$.

1.4. Suppose that

(a) there exists a sequence of disjoint measurable sets $(A_j)_{j \in \mathbb{N}}$ such that $\mu(A_j) \in (0, \infty)$ for all $j \in \mathbb{N}$ and $\sum_{j \in \mathbb{N}} \mu(A_j) = \infty$, and

(b) there exists a sequence of disjoint measurable sets $(B_k)_{k \in \mathbb{N}}$ such that $\mu(B_k) \in (0, \infty)$ for all $k \in \mathbb{N}$ and $\lim_{k \to \infty} \mu(B_k) = 0$.

Prove that for every $p \in [1, \infty]$ there exists $f_p \in L^p(\Omega, \mathrm{d}\mu)$ such that $f_p \notin L^q(\Omega, \mathrm{d}\mu)$ for any $q \in [1, \infty] \setminus \{p\}$.