Mathematisches Institut
LMU München
WS 2013-14

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Mathematical Quantum Mechanics

Homework Sheet 7

Exercise 1:

1. Prove the Fefferman–de la Llave formula: For $x, y \in \mathbb{R}^3$
$$\frac{1}{|x-y|} = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}R}{R^5}\int_{\mathbb{R}^3} \chi_{B_R(z)}(x)\chi_{B_R(z)}(y)\mathrm{d}z.$$
Here $\chi_\Omega(x)$ is the indicator function of Ω (i.e. $\chi(x) = 1$ for $x \in \Omega$ and $\chi(x) = 0$ otherwise).

2. Prove that
$$D(f,g) := \frac{1}{2}\iint \frac{\overline{f(x)}g(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y$$
is an inner product in
$$\mathfrak{C} := \big\{ f : \mathbb{R}^3 \to \mathbb{C} : f \text{ is measurable}, D\big(|f|,|f|\big) < \infty \big\}/\sim,$$
where $f \sim g$ iff $f = g$ almost everywhere. It follows that $\mathfrak{C}$ is a Hilbert space.

Exercise 2 (12 points): On the domain
$$\mathcal{D} := \big\{ \rho : \rho \in L^{5/3}(\mathbb{R}^3), \rho \geqslant 0, \nabla\rho^{1/2} \in L^2(\mathbb{R}^3), D(\rho,\rho) < \infty \big\}$$
consider the Thomas–Fermi–von Weizsäcker functional
$$\begin{aligned}\mathcal{E}(\rho) := \int \big(\nabla\rho^{1/2}(x)\big)^2\mathrm{d}x + \frac{3}{5}\left(\frac{6\pi^2}{q}\right)^{2/3}\int \rho(x)^{5/3}\mathrm{d}x \\ - Z\int\frac{\rho(x)}{|x|}\mathrm{d}x + \frac{1}{2}\iint\frac{\rho(x)\rho(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{1}$$

1. Prove that $\mathcal{E}$ has a minimizer $\rho_0 \in \mathcal{D}$.
2. Prove that $\mathcal{E}$ is strictly convex. Conclude that the minimizer is unique.
3. Derive the Euler equation for $\psi_0 := \rho_0^{1/2}$.
4. Find out, if $\rho_0 \in L^1(\mathbb{R}^3)$. In the latter case find an upper bound on $\int \rho_0(x)\mathrm{d}x$. *Hint:* Exercise 1.3 of Homework Sheet 2 may turn out to be useful here.

The solutions should be put to the box marked “Mathematical Quantum Mechanics” on the first floor **by 16:00 on Tuesday, Dezember 3**.

Every solution must be an original work of its single author!
Violations of this rule will be penalized!