Mathematisches Institut
LMU München
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Mathematical Quantum Mechanics

Homework Sheet 6

Exercise 1: Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis in $\mathfrak{H} := L^2(\mathbb{R}^3)$ consisting of functions from $C_0^2(\mathbb{R}^3)$. In the fermionic Fock space $\mathfrak{F}(\mathfrak{H})$ with the vacuum vector Ω consider the operator

$$H_0 := \sum_{j,k\in\mathbb{N}} \Big(e_j, \big(-\Delta - Z/|x|\big)e_k\Big) a^*(e_j)a(e_k) + \frac{1}{2}\sum_{j,k,l,m\in\mathbb{N}} \big(e_j\otimes e_k, |x-y|^{-1}(e_l\otimes e_m)\big) a^*(e_j)a^*(e_k)a(e_m)a(e_l)$$

with

$$D(H_0) := \mathrm{Span}\,\big\{a^*(e_{j_1})a^*(e_{j_2})\dots a^*(e_{j_K})\Omega : 1 \leqslant j_1 < j_2 < \dots < j_K, K\in\mathbb{N}_0\big\}.$$

1. Show that H_0 is a densely defined symmetric operator in $\mathfrak{F}(\mathfrak{H})$.
2. Prove that there exists a self-adjoint extension H of H_0 with a form domain contained in the form domain of the particle number operator $N = \sum_{j\in\mathbb{N}} a^*(e_j)a(e_j)$.
   Hint: Find a constant C such that $H_0 + CN$ is bounded below.
3. Show that for every $K \in \mathbb{N}_0$ the subspace $\mathfrak{H}_K$ (i.e., the K-particle sector of $\mathfrak{F}(\mathfrak{H})$) is invariant for H. Describe the action of H in this subspace not using creation/annihilation operators.
4. Conclude that H does not depend on the choice of the basis $\{e_j\}_{j\in\mathbb{N}}$.

Exercise 2: For a system of K identical fermions with spin 1/2 (in particular, electrons) the natural Hilbert space is $\wedge^K\mathfrak{H}$ with $\mathfrak{H} = L^2(\mathbb{R}^3,\mathbb{C}^2)$. Show that the ground state energy of the atomic Hamiltonian

$$H = \sum_{j=1}^{K}\Big(-\Delta_j - \frac{Z}{|x_j|}\Big) + \sum_{\substack{j,k=1\\ j<k}}^{K} \frac{1}{|x_j - x_k|}$$

is bounded below by

$$-\frac{Z^2}{2}(M-1) - \frac{Z^2}{4M^2}\Big(K - \sum_{n=1}^{M-1} 2n^2\Big), \text{ where } M := \min\Big\{N\in\mathbb{N} : \sum_{n=1}^{N} 2n^2 \geqslant K\Big\}.$$

Continues on the next page!

You may use the spectral decomposition of the one particle Hamiltonian $-\Delta - Z/|x|$ without proof.

Exercise 3: Determine the essential spectrum of $-\Delta - \left(1 + |x|^2\right)^{-1/4}$.
Hint: You may use Seiler-Simon inequality.

The solutions should be put to the box marked "Mathematical Quantum Mechanics" on the first floor **by 16:00 on Tuesday, November 26**.

Every solution must be an original work of its single author!
Violations of this rule will be penalized!