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Mathematical Quantum Mechanics

Homework Sheet 3

Exercise 1 (2 points): Assume that the Born–Jordan commutation relation

$$[p, q] = -\mathrm{i}\hbar$$

is satisfied by bounded operators p and q. Conclude that the Planck constant must be zero, i.e., there is no quantum mechanics.

Exercise 2: In this exercise on simple examples we explore why defining an observable on a natural dense set is not sufficient (contrary to a common belief!) and how much the properties of an operator depend on its domain.

In the Hilbert space $L^2\big((-1,1)\big)$ let the operator A_0 be defined on $D(A_0) := C_0^\infty\big((-1,1)\big)$ by the formula

$$(A_0 u)(t) := -\mathrm{i}\frac{\mathrm{d}u(t)}{\mathrm{d}t}.$$

1. Show that A_0 is symmetric, but not selfadjoint. Find all selfadjoint extensions of A_0, i.e., such operators $A = A^*$ that $D(A_0) \subset D(A)$ and $Au = A_0 u$ for all $u \in D(A_0)$.
   Hint: The crucial observation is that every symmetric extension A of A_0 must satisfy $D(A_0) \subset D(A) \subseteq D(A^*) \subset D(A_0^*)$ and $A = A_0^*|_{D(A)}$.
2. Determine the spectra of these extensions.
3. Find a (non–selfadjoint!) extension of A_0 whose spectrum is empty.
4. Replace the interval $(-1,1)$ by $(0,\infty)$ and find the selfadjoint extensions of A_0 in this case.

Exercise 3: For $\phi \in \mathcal{S}(\mathbb{R}^3)$ consider

$$E_Z[\phi] := \int_{\mathbb{R}^3} \sqrt{|\xi|^2+1}\,\big|\phi(\xi)\big|^2 \mathrm{d}\xi - \frac{\alpha Z}{2\pi^2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\overline{\phi(\xi)}\phi(\eta)}{|\xi-\eta|^2} \mathrm{d}\eta \mathrm{d}\xi \qquad (1)$$

1. Find the biggest value Z_c of Z, for which $E_Z[\phi]$ is nonnegative.
   Hint: Reduce the problem to the one with $\sqrt{|\xi|^2+1}$ replaced by $|\xi|$. This can be solved by using the ground state transform with the ground state function $|\xi|^{-2}$.

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2. For every function $f \in \mathcal{S}(\mathbb{R}^3)$ calculate the ratio

$$A := \left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\overline{\hat{f}(\xi)} \hat{f}(\eta)}{|\xi - \eta|^2} \mathrm{d}\xi \mathrm{d}\eta \right) \Big/ \left(\int_{\mathbb{R}^3} \frac{\left|f(x)\right|^2}{|x|} \mathrm{d}x \right).$$

3. Calculate the quadratic form q_C of $C = \sqrt{-\Delta + 1} - \alpha Z/|x|$ in terms of E_Z.

4. Assuming $Z \leq Z_c$, determine whether there is a unique selfadjoint operator C with $D(C) \supset \mathcal{S}(\mathbb{R}^3)$ such that its quadratic form is given by q_C.

The solutions should be put to the box marked "Mathematical Quantum Mechanics" on the first floor **by 16:00 on Tuesday, November 5**.

Every solution must be an original work of its single author!
Violations of this rule will be penalized!