Exercise sheet 3

Exercise 1.

Let A be a commutative ring and M be a finitely presented A-module.

1) Let n be an integer. Show that if $\phi : A^n \twoheadrightarrow M$ is an epimorphism of A-modules, the A-module $Ker(\phi)$ is of finite type.

2) Show that if M is projective, it is a summand of a free A-module of finite type.

3) Show that (M is flat) $\Leftrightarrow$ (M is a projective module of finite type) [Hint: you may use that M is a filtering colimit of finite type free A-modules]

4) If M is flat, show that it is locally free of finite type, that is to say, $\exists (f_1, \dots, f_r) \in A^r$ such that the ideal $(f_1, \dots, f_r)$ in A is A itself, and for each i, $M_{(f_i)}$ is a free $A_{(f_i)}$-module of finite type.

Exercise 2.

1) Let $\phi : A \to B$ be a morphism of commutative rings, with B a free A-module of finite type. Show that the morphism $\Phi := Spec(\phi) : Spec(B) \to Spec(A)$ is open. For that show that for any $f \in B$, $\phi(D_B(f))$ is open in $Spec(A)$. [Hint: use the characteristic polynomial of $f \in B$ over A]

2) Show that a finite, flat morphism $Y \to X$ between noetherian schemes is open. [Hint: reduce to the case X (and Y) are affine, and use exercise 1) to reduce to the case $X = Spec(A)$ and $Y = Spec(B)$ with B a free A module of finite type]

Exercise 3.

Show that a dominant morphism $Y \to X$ of k-schemes with X and Y integral schemes of finite type over a field k and X a regular curve over k is automatically flat. [Hint: use the structure of finite type modules over a P.I.D.....]