Exercise sheet 2

Exercise 1.

Denote by $\mathcal{F} \subset Set$ the full subcategory of finite sets. Show that a functor $F : \mathcal{F} \to Set$ which takes finite limits to finite limits is pro-representable. Let $S \in Set$ be a set; how would you define its profinite completion $\hat{S}$?

Denote by $\mathcal{FG}r \subset \mathcal{G}r$ the full subcategory of the category of groups consisting of finite groups. Show that a functor $F : \mathcal{FG}r \to Set$ which takes finite limits to finite limits is pro-representable. Let $G \in \mathcal{G}r$ be a group ; how would you define its profinite completion $\hat{G}$?

Give an example!

Exercise 2.

Let $(\mathcal{C}, \mathcal{T})$ be a site endowed with a Grothendieck topology. Let P be a presheaf of sets on $\mathcal{C}$. Let $\mathcal{C}/P$ be the category whose objects are pairs (C, x) with C an object of $\mathcal{C}$ and $x \in P(X)$ and whose morphisms $(C, x) \to (D, y)$ are morphisms $\phi : C \to D$ in $\mathcal{C}$ such that $P(\phi)(y) = x$. We have the obvious (forgetful) functor $(\mathcal{C}/P)^{op} \to Pshv(\mathcal{C})$,$(C, x) \mapsto C$ (by which we mean $C \in PShv(\mathcal{C})$ is the presheaf $Hom_{\mathcal{C}}(-, C)$ represented by C). Show that the obvious morphism:

$$colim_{(C,x)\in\mathcal{C}/P} C \to P$$

is an isomorphism.

Exercise 3.

Let $(\mathcal{C}, \mathcal{T})$ be a site endowed with a Grothendieck topology. Let $f : F \to G$ be a morphism in $Shv_{\mathcal{T}}(\mathcal{C})$.

Show that :

(f is an epimorphism) <=> (f is an effective epimorphism) <=> (f is a universal effective epimorphism)

[Hint: consider the diagram $F \times_G F \rightrightarrows F \to G$ also in the category of presheaves and use a remark of the lecture concerning monomorphisms of presheaves...]