On the relation beween the Efimov effect in Quantum Mechanics
and the Thomas effect in Nuclear Physics.

(the role of model in Physics)

Dedicated to Beppe Marmo, on the occasion of the celebration of his 65^{th} birthday

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1) INTRODUCTION

In this short review we discuss the relation between the *Efimov effect* in Quantum Mechanics and the *Thomas effect* in nuclear physics.

The Efimov effect is the following property: if a three-particle system with two-body forces is such that all the two-body subsystems have positive spectrum and if at least two of them have a zero-energy resonance, then the three-body system has an infinite number of negative eigenvalues accumulating at zero.

Recall that a zero energy resonance for a two particle system with hamiltonian H is a solution of $H\phi = 0$ that is not square integrable but satisfies $\int |\phi(x)|^2 (1+|x|)^{-k} d^3x < \infty$ for k large enough.

This remarkable spectral property was described theoretically by V.Efimov [E] and has since become the subject of many papers. The first mathematical proof of the Efimov effect was given by Yafaev [Y].

A.Sobolev [S] estabilished the following asymptotic for the function $N(z), z < 0$, the number of eigenvalues below z

$$lim_{z\to 0}|log|z|^{-1}N(z) = C \qquad 1.1$$

where C is a constant that depends on the mass of the particles *but not on the potentials.* See also [Ta1][TA2]].

The bound states with energy nearer the threshold have a wider support (in some sense, the accumulation is an infrared phenomenon).

It can be shown [W] that an Efimov-like effect is also present if instead of a three partice system one has a N particle system in which the conditions we have assumed for the three particle system are satisfied in three channels. In this case case the properties of the spectrum have not been fully clarified yet.

We will come back to the Efimov effect and sketch the ingredients of its proof.

A similar effect, the Thomas effect, in known in Nuclear Physics.

This effect goes back to an attempt by Thomas [T] (whence the name) to describe the three-body problem in nuclar physics by taking the forces to be *of zero range,* as did Fermi in the analysis of neutron scattering [F].

Nuclear forces are of very small range, therefore the model seems appropriate.

For this model not only there are infinitely many bound states for a three body system, but the spectrum is unbounded below with eigenfunctions which tend to have support in a region which shrinks to a point (of total collapse).

The first theoretical analysis of the Thomas effect can be found in [ST].
The first rigorous proof was given by Faddeev and Minlos [MF1][MF2] who use the mathematical meaning of "zero range interaction" derived from the theory of self-adjioint extensions.
In their papers Minlos and Faddaev constructed a family of self-adjont isospectral operators (unbounded below) for a system of three particles interacting through zero range interaction.
They proved also an asymptotic formula for the ratio of the energy of two successive eigenvalues which is similar to the asymptotic formula derived by Sobolev and Tamura for the Efimov effect.
Later the unboundedness of the spectrum was also proved with another technique by [DFT] who described the construction of N-body zero range interactions using quadratic (energy) forms
The relation between the two effects was noted by Faddaev who suggested to Yafaev [Ya] to study the connection between the two. Tis led eventually to the mathematical proof of the Efimov effect.

In experiments in nuclear physics the Efimov-Thomas effect is described (see e.g. the review by Hammer [H]) as the appearance of many bound states in a three-body system with a large two-body scattering length, with distribution of energies which shows some sort of universality.
This is regarded as remnant of an Efimov-like effect that appears in the limit of infinite scattering length (unitary limit) [BH[HP] .
The Efimov-Thomas effect in Nuclear Physics has recently received a significant of attention as a consequence of major advances in experimental techniques, that allow a tuning the magnetic field to achieve a zero energy Feshbach resonance .
The recently described bound state of the system *pnn* with total spin 1 is regarded as the ground state of an Efimov system, and the discovery of at least another bound states in this system have been reported.
A promising system for observing several Efimov states (called *trimers)* is the system composed of He^4 atoms which have a large scattering lenght.
In this systems experimenters have also searched for *tetramers* (bound states of four atoms). Their presence can be also predicted by models, since it is enough to prove that the lower bound of the spectrum for a four particle system lies below the ground state for the particle system.
The universal character of the low energy asymptotic (which depends on the masses but not on the detailed form of the potential) has prompted also attempts to make use methods typical of the renormalization group.
In the case of zero range interactions, if two of the particles are identical fermions (so that there is no interaction between them) the lower bound of the spectrum is finite if the mass of the third particle is sufficiently large as compared to the mass of the fermions and a genuine Efimov effect is present. This was proved by R.Minlos [M] with later improvements in Minlos and Shermatov [MS] and Shermatov[S].
The Efimov effect has been studied also in lattice models with a special form the "kinetic energy" . in [ADL] and under more general conditions in [DMS]. It is shown that the Efimov effect is present when the system is translation invariant and that if two of the particles are identical fermions the Efimov effect has a different threshold.
The striking similarities between the Efimov and Thomas effects has led in the physics literature to call both "Efimov effect" and in the mathematical literature to a suggestion

(see e.g. [A] and [MM*]) that there is some relation between the two formalisms, in spite of the fact that one (Efimov) is a "long range effect" (infrared) and the other (Thomas) is a "collapse to a point" ("ultraviolet)".

There are evident similarities: in order to have the Efimov effect there must be a zero energy resonance in at least two of the three channels, i.e. a distinguished element which does not belong to the Hilbert space of the square integrable functions of the channel.

Similarly , as we shall see, in the Thomas effect in at least two of the channels there is a distinguished element in the domain of the hamiltonian operator. This element can be taken to be the Green function of the Laplacian, which is square integrable only locally i.e satisfies the criterion usually given for a resonance .

Remark 1

We close this introduction pointing out the relevance of the study of exactly solvable models in which exact formulae are given, e.g. for the number of bound states and their energies.

Simple solvable modes can give hints to what is reasonable to expect in more realistic cases in which naive perturbative arguments are going to fail, as is the case in nuclear physics for strong short range forces in presence of resonances.

Models are essential in studying non perturbative effects, Zero energy resonances are very fragile objects. A very small change in the potential could lead either to a loosely bound states or to a continuum spectrum with regular r spectral weight at the origin.

On the other hand simple minimax arguments show that i a model has N bound states with energies $E_k < B,\ k = 1, ..N$ and if one adds a potential V bonded uniformly by a constant c ($|V(x)| \leq c$) the resulting system will have at least N bound states with energy less that $B - c$

Therefore addition of a small potential does not alter the number of bound states which are *sufficiently bound* and moroever the ratio between the energies of successive eigenvalues are only slightly changed if the energies are "very negative".

On the other hand, zero-range interaction are an idealization of realistic short range forces. Since the bound states for a Schroedinger system with hamiltonian H equation are the singularities on the negative real axis of the resolvent $(H-z)^{-1}$, a control of the distribution of bound states in a realistic system can be achieved by proving the "proximity" in same sense of the resolvents

In this way the Thomas effect seen in Nuclear Physics can be seen as a "remnant" of an Efimov effect that would hold under idealized condtions.

♣

Remark 2

The exact formulae which can be given for the point interaction of N particles may suggest to look within this model for a possible presence of at least one bound state for a system of four nucleons (tetramer) for values of the parameters for which the bottom of the spectrum of the three body system is finite.

Models can be effective also to determine the presence of a four particles bound state (tetramer) (at least one possible tetramer has been reported [CMP]).

In the model a tetramer exists if the lower bound of the spectrum of a four particle system is lover than the ground state of the three particle one.

♣

2) Zero range interaction as limit of an interaction with a zero energy resonance

In this chapter we shall study further the relation between the Thomas effect and of the Efimov effect.
It is convenient to start by describing in R^3 a two-body system with zero range interaction as limit of a two body system with a zero energy resonance and potential with very short range.
In this model the hamiltonian H_α is a self-adjoint extension of the symmetric operator $-\Delta$ acting on functions that have compact support , are in the Sobolev space $\mathcal{H}_2(R^3)$ and vanish in a neighborhood of the origin in R^3.
It follows from Krein's theory that the domain of any such extension are the function $\phi(x) \in L^2(R^3)$ which can be written as

$$\phi(x) = \frac{c}{|x|^2} + \psi(x), \qquad \psi \in \mathcal{H}_2(R^3) \qquad 2.1$$

supplemented by the relation

$$\psi(0) = \alpha c \qquad 2.2$$

The parameter α, which qualifies the extension, has the interpretation of inverse of the scattering length and the unitary limit corresponds therefore to $\alpha = 0$.
The operator H_α acts on its domain as

$$(H_\alpha \phi)(x) = -\Delta\psi \qquad 2.3$$

The energy form (expectation value of the hamiltonian) can be written if $\alpha \neq 0$as

$$Q(\phi) \equiv (\psi, H\psi)(x) = -(\nabla\psi, \nabla\psi)(x) + \alpha|\psi(0)|^2 \qquad 2.4$$

In the case $\alpha = 0$ one has $\psi(0) = 0$ and c^2 takes the place of α in (2,4).
If one substitutes (2.1), (2.2) in (2.3) one derives *formally* that H represents the interaction with a potential *concentrated at the origin.*
This interpretation, mathematically imprecise, may suggest , from the physical point of view, that the zero range interaction may be obtained from a short range potential with a limiting procedure. In this case the exact results for zero range interaction are a valid approximation for potentials of very short range.
Another approximation by exact solvable models can be made by resorting to a lattice model using a potential which is concentrated in a point .
The Efimov effect for the lattice case has been analyzed in [ALT],[DMS] but the limit has not been studied so far.
In the case of the continuum the comparison can be done using the following result [AHK] which shows that for a two particle system the zero range interaction in R^3 can be approximated in the strong resolvent sense by an interaction through a regular potential that has a zero energy resonance.
Recall that the resolvent is a natural instrument to be used in locating bound states, as they are singularities of the resolvent.
We briefly sketch the main steps, mainly to introduce the instruments that are later used in the proof of the Efimov effect. In particular we want to stress that an important role in determining the number and distribution of the bound states below $-a^2$ is played by the Birman-Schwinger operator $\sqrt{|V|}(-\Delta + a^2)^{-1}\sqrt{|V|}$

Let

$$H(\lambda) = -\Delta + V_\epsilon \quad V_\epsilon(x) = \frac{\lambda(\epsilon)}{\epsilon^2} V(\frac{x}{\epsilon}) \quad \lambda(0) = 1 \tag{2.5}$$

For simplicity we consider only that case $V \leq 0$.
With

$$v(x) = \sqrt{-V(x)} \qquad G_k = \frac{1}{-\Delta - k^2}$$

one has

$$\frac{1}{-\Delta + V - k^2} = G_k + (G_k v)\frac{1}{1 - vG_k v}(vG_k) =$$

$$G_k + A_\epsilon(k)\frac{\lambda(\epsilon)\epsilon}{1 + B_\epsilon} C_\epsilon(k) \tag{2.6}$$

where $A_\epsilon,\ B_\epsilon\ C_\epsilon$ are Hilbert Schmidt operators with the following behavior for $\epsilon \to 0$

$$A_\epsilon(k) \to |G_k >< v|, \quad B_\epsilon(k) \to -vG_0 v \quad C_\epsilon \to -|v >< G_k|$$

In fact

$$B_\epsilon(k) = -(vG_0 v)(k) - \epsilon\lambda'(0)(vG_0)v(k)) - \frac{ik}{4\pi}|v >< v| + o(\epsilon)$$

It is easy to see that, if there exists $\phi \in L^2$ which satisfies

$$\phi \in L^2, \ \ vG_0 v)\phi = \phi \tag{2.7}$$

then

$$H\psi = 0, \qquad \psi(x) \equiv (G_0 v)\phi)(x)$$

and if $< v, \psi > \neq 0$ then ψ is not in L^2 but satisfies, for p large enough

$$\int |\psi(x)| \frac{1}{(|x^2| + 1)^p} dx < \infty$$

(i.e. it represents a resonance).
If there is a resonance one has

$$\frac{1}{1 - vG_0 v + z} = \frac{1}{z} P + regular \qquad P(x, y) = \phi(x)\phi(y) \tag{2.8}$$

and

$$\frac{\epsilon}{1 + \epsilon + B_0(k)} = P + o(\epsilon) \tag{2.9}$$

Therefore

$$lim_{\epsilon \to 0} \frac{1}{H(\epsilon) - k^2} = \frac{1}{-\Delta - k^2} + |G_k > \frac{1}{\frac{ik}{4\pi} + \frac{\lambda'(0)}{|<v|\phi>|^2}} < G_k| \tag{2.10}$$

This is the resolvent of H_α where $\alpha = \frac{\lambda'(0)}{|<v|\phi>|^2}$ is the ratio of the regular part and the coefficient of the singular part of the elements in the domain.

We conclude that one has resolvent convergence of $H(\lambda(\epsilon))$ to H_α where $\alpha = \dot{\lambda}(0)$.
In particular in the limit there is no negative eigenstate (but a zero energy resonance) *only if* $\alpha = 0$ and therefore only in this case the zero range interaction will be effective in comparing Efimov's and Thomas' effect.

Notice that the operator V_ϵ converges weakly to zero (but it does not converge strongly) . As a consequence the operators $H(\lambda(\epsilon)$ converge weakly to $-\Delta$ *and not to*H_α. Indeed the scaling we have made of the peak of the potential is by a factor ϵ^{-2} whereas to obtain as a limit a delta function the scaling would have been by a factor ϵ^{-3}.
Still, if ϕ_ϵ is the function which represents the resonance,

$$(\phi_\epsilon, V_\epsilon \phi_\epsilon) = 1 \qquad \forall \epsilon$$

by our choice of normalization.
The effect of the resonance survives the limit $\epsilon \to 0$.
The limit in resolvent sense is H_α only because there is a zero energy resonance and this leads to an extra term in the low energy structure of the resolvent (an infrared effect).
On the other hand, resolvent convergence implies convergence of the wave operator and therefore of the scattering matrix.

Still, the limit of the energy form of $H(\lambda(\epsilon)$ is related to the energy form of H_α. modulo a "renormalization". This is typical of a weaker form of convergence, Γ-convergence, that can also be used to define zero-range interaction [DFT].
A a last remark, we notice that information obtained from the model for some value of the parameters that enter the interaction Hamiltonian may be relevant also for the physical system for slightly different values of the parameter

3) Three particle point interaction; zero-range interaction case

We have seen that in a two-body system a good model to describe the low-energy (long distance) limit of a short range interaction which exibits a resonance is a model in which the interaction is introduced exclusively through a resonance (i.e. extending the free laplacian by adding a resonance to its domain).
We extend now this reasoning to the resulting N body system. The case $N = 3$ will lead us to an analysis of the Efimov effect and its relation with the Thomas effect.
We are interested in bound states of a Nbody system which lie below the bottom of the continuous spectrum of the $N - 1$ body system (there may be bound states imbedded in the continuous, but this is an exceptional case, due in general to superselection rules).
Therefore relevant informations on the Nbody system may be obtained only if the energy of the $N - 1$ body system is bounded below.
This is the case for $N = 3$, the case we analyze now.
For values of the parameters for which if $N = 3$ the energy spectrum in unbounded below, no information follows for the case $N \geq 4$.
Therefore the use of the model to obtain estimates for the four-particel case is restricted to situations under which the model for $N = 3$ has an energy spectrum which is bounded below.

We start constructing an energy form associated to a system o three particles intreracting through zero range interactions. We shall proceed in analogy with the two-body case; later we will construct a self-adjoint operator which has it as energy form.

The most natural way is to use Krein's theory of self-adjoint extensions starting from the symmetric operator $-\Delta_0$, the free laplacian in R^9 defined on H^2 functions functions of compact support which vanish in a neighborhood of the (coincidence) set

$$\Gamma \equiv \cup \Gamma_{i,j} \; i = 1,2,3, \; i \neq j \qquad \Gamma_{i,j} = \{x_i = x_j, \; x_i \in R^3$$

This operator is symmetric but not self-adjoint and according to Krein's theory its extension are classified by the solutions of the equation

$$(-\Delta_0 \pm i)\psi = 0 \quad x \in R^9$$

We are interested in a subclass of extensions, and in particular those that can be obtained as limit in resolvent sense of hamiltonians H_{V_ϵ} which are translation invariant and satisfy the conditions to produce the Efimov effect, since we want to find the relation between this effect and Thomas'.
Such extensions can be constructed, in analogy with what happens for a two particle system, by adding to the domain of Δ_0 functions of the form

$$f(Y)G_Y(x - Y) + g(x) \quad Y \in \Gamma \qquad 3.1$$

where G is the Green function of the Laplacian which vanish at infinity and f, g are functions of suitable class with with g continuos at Γ.
Notice that these functions are not in the domain of Δ_0. Since the domain is a linear space, such extensions are classified by giving a linear relation between $f(Y)$ and $g(Y)$ (recall that $g(x)$is continuous) i.e. by an operator K on Γ such that $G = KF$.
The domain of the self-adjoint extension is made of function of the form

$$\phi(x) = \psi(x) + \int_\Gamma \int f(y)G_Y(x - y)dy + g(x) \quad g(y) = (Kf)(y) \quad y \in \Gamma \quad \psi \in D(H_0) \qquad 3.2$$

In analogy with the case of two particles, we expect that the limit of a system with no two body bound states and a zero energy resonance in at least two channels correspond to the extensions for which $g(x) = 0$ on Γ.
If we want to consider a system composed of two identical fermions and another particle we must choose $f = 0$ on the hyperplane $x_1 = x_2$ and suitably antisymmetric for the exchange $x_1 \leftrightarrow x_2$. If one wants a system which is translation invariant it is convenient to choose Jacobi (barycenter) coordinates.
It is easy to see that the bilinear form $Q(\phi) = (\phi, H\phi)$ associated to any extension obtained in this way has the expression

$$Q(\phi) = \int |\nabla\psi|^2 dx + \hat{Q}(f) \qquad \hat{Q}(f) = \hat{Q}_{diag}(f) + \hat{Q}_{off}(f) \qquad 3.3$$

with The weight function in $\hat{Q}_{diag}$ comes from the use of Jacobi coordinates and integration of the Green function of Δ in R^3. Since this integral is marginally divergent, one uses the Green function of $-\Delta + \epsilon$ and adds a term $\epsilon|\psi|_2^2)$.
The expressions we shall obtain have a limit when $\epsilon \to 0$. Therefore, abuse of notations, we will simplify te formulas by taking $\epsilon = 0$

The (negative) bound states are determined studying the negative part of the quadratic form $\hat{Q}$ and the associated operators. We shall therefore neglect the modification one should do by adding the parameter ϵ. This is relevant only in the study of the low energy limit of the continuum spectrum of the extension.
In the case of two identical fermions of mass 1 and another particle of mass m there is m^* such if $m < m^*$ the energy form is unbounded both above and below and it is not closed on its natural domain.
The sequence of test functions along which the form takes values which are more and more negative is made of functions on R^3 (which we can take to be the hyperplane $x_1 = x_3$) which are uniformly in L^2 but with diverging $H^{\frac{1}{2}}$ norm.
This comes form the structure of $\hat{Q}_{off}$ which is bounded in $H^{\frac{1}{2}}$ but not in L^2; for $m < m^*$ its negative part it dominates the (positive) term $\hat{Q}_{diag}$ and the sum of the two terms has a negative part which diverges to ∞ along the sequence chosen.
For $m < m^*$ (which is the case in the cases considered in nuclear physics) the form $\hat{Q}$ cannot be associated to a unique operator K . One has to choose a particular subset in the domain of the form on which the form is closed and is associated to a self-adjoint operator which is of course unbounded both above and below.
A one parameter family of operators K_α have been constructed in [MF1] with form coinciding with Q on their domain. The construction is made by constructing operators K_α associated to the form $\hat{Q}$ and therefore defined on a dense domain in $L^2(R^3$.
All the operators constructed in this way have discrete spectrum and *are isospectral.*
The sequence of eigenfunctions which correspond to eigenvalue diverging to $-\infty$ have Fourier transforms which "concentrates at infinity" along a fixed direction in R^3. Different selfadjoint extensions are obtained by rotation in R^3.
Recall that the negative spectrum of any selfadjoint extension of H_0 is identical with the negative spectrum of the operator K_α (which, as remarked, is independent of α).
By its very construction the operator K_α are Toeplitz operators.
Rougly speaking, a Teoplitz operator is an operator which, written in a suitable basis, has matrix element which satisfy suitable recursion relations. As a consequence the eigenvalues of Toepltz operators satisfy recursion relations, of the type of those which lead to the asymptotic structure in the Efimov effect.
In the case of zero range interactions these recursion relations have a particular "pure" form and in particular Minlos and Faddaev in [MF2] have proven that the operators K_α can be diagonalized by a Mellin transform. This leads to an exact formula for the ratio $\gamma(m)$ of successive eigenvalues (that obviously depends only on the ratio of the masses of the three particles) and to an explicit determination of the eigenfunctions.
This result holds true in particular in the case of a system of two identical fermions of mass 1 interacting with a different particle of mass m.
In this case Minlos and Shermatov [MS[MS][S] have proven that there is a constant m^* such that if $m > m^*$ the quadratic form $\hat{Q}$ is bounded below and closable. It is therefore associated to a unique selfadjoint operator K which turns o to be compact. It has infinitely many eigenvalues with constant constant ratio between successive eigenvalues.
For $m < m^*$ the quadratic form is unbounded below, any selfadjoint extension is unbounded below and no conclusion can be obtained about the bound states; if they exists, they are embedded in the continuum spectrum.
In the case $m > m^*$ it is interesting the study of a system obtained by adding one more identical fermion, i.e. to study a system of three identical fermions of mass one in interac-

tion, through zero range interaction, with a fourth particle of mass $m > m^*$
In this case preliminary estimates show that the lower bound of the spectrum for this systems is below the energy E_3^0 of the ground state of the three body system (which is finite and strictly negative).
If this is the case, there exist at least one bound state of the system, and possibly many (the spectrum of the 4-body system above E_3^0 is continuum).
The advantage of considering this limit situation is that the results one obtains , in particular about the spectrum of the bound states, are exact , and can give a reasonable hint to the distribution of bound states in more realistic situations, even for mass ration smaller that m^*.
We will give details only for a three particle system in which two of the particles are identical fermions (therefore there is no point interaction between them) of mass 1 interact through point interaction with a third particle of mass m.
It is convenient to work in momentum space, after having factor out the free motion of the center of mass.
The Hamiltonian H_0 is then presented as multiplication by $p_1^2 + p_2^2 + \mu p_1.p_2$ on functions $f \in L_a^2(R^6)$ (the suffix a stands for antisymmetric) which satisfy

$$\int (p_1^2 + p_2^2)|f(p1, p_2)|^2 dp_1 dp_2 < \infty, \int f(p_1, p_2) dp_k = 0, \quad k = 1, 2 \qquad 3.4$$

The parameter $\mu = \frac{2}{1+m}$ is the reduced mass.
The deficiency subspace of h (real part of the solutions of $(h^* - i)\phi = 0$) consists of elements of the form

$$\Phi_u = \frac{u(p_1) - u(p_2)}{p_1^2 + p_2^2 + \mu p_1.p_2} \qquad 3.5$$

where $u(p)$ satisfies

$$\int |u(p)|^2 (p^2 + \lambda)^{-1over2} dp < \infty \qquad 3.6$$

Krein's theory for semi-bounded symmetric operators can be applied with the result that all self-adjoint extension of h can be described. For their description it is convenient to introduce the Hilbert space $\mathcal{K}$ of functions on R^3 which is the completion of $L^2(R^3)$ with respect to the scalar product $< u, v >= (Ku, v)$ where

$$(Ku)(p) = \frac{4\pi^2}{\sqrt{(4-\mu^2)p^2 + 4}} u(p) - 2 \int \frac{u(k)}{p^2 + k^2 + \mu p.k + \lambda} dk \qquad 3.7$$

With these notations the self-adjoint extensions are in one-to-one correspondence selfadjoint operators A on $\mathcal{K}$ and the domain of the self-adoint extension associated to A is

$$D(H_A) = \{g \in D(H^*) \ : \ g = f + \Phi_u + \hat{h} + \lambda^{-1} \Phi_v, \quad f \in D(h), \quad u \in D(A) \quad Av = u \qquad 3.8$$

where $\hat{h}$ is the free extension of h.
In analogy with the two-particle case we call the function v (defined on Γ) "charge". The operator H_A acts on its domain as

$$(h_A g)(p_1, p_2) = ((p_1^2 + p_2^2 + \mu p_1.p_2) f + [v(p_1) - v(p_2)](\frac{1}{p_1^2 + p_2^2 + \mu p_1.p_2) + \lambda} + \Xi(\lambda) \qquad 3.9$$

where $\Xi(\lambda)$ is is a bounded operator which vanish for $\lambda = 0$ (the parameter λ is introduced because h is not invertible).
In the search of bond states with strictly negative energies we can safely set $\lambda = 0$; this choice simplifies considerably the formulas.
The formula one obtains for the quadratic form Q_N in the general case of N fermions of mass one interacting via zero range forces with a different particle of mass m is more easily expressed using the Fourier transformed function $\hat{u}$

$$Q_N(u) = Q_{N,diag}(u) + Q_{N,off}(u) \qquad 3.10$$

where

$$Q_{N,diag}(u) = 2\pi^2 \int_{R^N} |\hat{u}|^2(p_1,..,p_N)[\frac{m(m+2)}{(m+1)^2} \sum p_k^2 + \frac{2m}{(m+1)^2} \sum_{k_i \neq k_j} p_i.p_j]^{\frac{1}{2}} \quad p_i \in R^3$$

$$Q_{N,off}(u) = N \int_{R^{3N+3}} dsdtdp_2...dp_N \hat{u}^*(s,p_2..p_N)\hat{G}(s,t,p_2,..,p_N)\hat{(u^*)}(t,p_2,..,p_N) \quad 3.11$$

where $\hat{G}$ is the Green function of the laplacian in $R^{3(N-1}$ written in Fourier coordinates and $\hat{u}$ is totally antisymmetric in its variables.
As in the case $N = 2$ the energy of the bound states of the system of $N+1$ fermions, $N \geq 2$ are the same a those of the operator H_{N+1} associated to the quadratic form $\hat{K}(N+1)$ which lie below E_N^0, the energy of ground state of the operator H_N. Recall that $E^0(N)$ is the onset of the continuum spectrum of H_{N+1}).
If the latter is unbounded below ($E^0(N) = \infty$), nothing can be said about the discrete spectrum of H_{N+1} (there could be eigenvalues embedded in the continous spectrum).

4. The Thomas effect as limit case of the Efimov effect

To compare this result for zero-range interaction with the Efimov effect, it is convenient to study the proof of this effect as given by Sobolev and to replicate it for a sequence of potentials, of fast decreasing range, for which the two-body hamiltonian has positive spectrum and a zero energy resonance (as in the proof by Sobolev) .
In this way we could prove resolvent convergence (as in the two particle case) and this would prove that the Thomas efffect is a limit case of the Efimov effect.
As for to the two particle case, this would be achieved if one would prove that the main ingredients that enter the proof, and that are the dominant terms in the convergence analysis in low energy limit, each converge in the limit to the main ingredients of the proof that we have given for the Thomas effect.
In the case we are interested in we can take the potential to be $V = \sum_i V_i$ where V_i is the potential between the i^{th} electron and the other particle (i.e the electrons interact among themselves only through their common interaction with the other particle). This simplifies the estimates.
We will not give the details, and limit ourselves to recall that these main ingredients of the case of the potential are (considering only negative potentials and using the (Birman) notation $v(x) = \sqrt{-V(x)}$

1)
the kernel

$$W_{i,j}(z) = (1 + v_i R_0(z) v_j)^{-1}, \quad i = 1,..N$$

(in our case the terms on the diagonal are absent)

1)

the matrix-valued kernel

$$A(z) = |\mathcal{W}(\ddagger)|^{\frac{1}{2}}\mathcal{K}(z)|\mathcal{W}(\ddagger)|^{\frac{1}{2}}, \quad K_{\alpha,\beta} = (1+v_\alpha)(1+v_\beta)$$

3)

the Birman kernels

$$G_0^{h,k}(x,y) = \frac{m}{2\pi}v_h(x)\frac{1}{|x-y|}v_k(y) \quad G_1^{h,k}(x,y) = \frac{m}{2\pi}v_h(x)v_k(y)$$

which enter the formulae that we have given above for the three body problem of point interaction.
As in the two particles case, scaling the square root v of $-V(x)$ according to $v_\epsilon(x) = \epsilon^{-1}v(\epsilon x)$ the expectation value of the the resolvent of the resulting operator $H(V_\epsilon)$ in a generic state is dominated, in the low energy, by terms which in a Born series expansion come from $v\psi$ when ψ represent the resonance.
Recall that $\phi \equiv v\psi \in L^2$ so one can decompose the two-particle Hilbert space $\mathcal{H}_2$ using the projection operator P_ϕ. The projections obtained in the different "channels" do not commute, and this leads to a reduction of the Born series to a collection of terms which come from resonances in different channels plus a residual collection of terms each of which converges to zero in the limit $\epsilon \to 0$ (recall that the potentials convergence weakly to zero in the limit, due to the choice of scaling). The difficult part of the proof is to show the sum of this second collection of terms converges to zero in te limit. The terms of first collection, i.e. those that come by from the presence of the resonances in at least two channels, can be partially resummed and the lead to the Born expansion of the resolvent of H_α..
The error in the approximation is $C(E,\phi)o(\epsilon$ with a coefficient C that depends on the state ϕ considered and on the energy considered. Due to the scaling, there is a positive constant $d(\phi)$ $C(E,\phi$ is bounded uniformly for $E < \frac{d}{\epsilon}$.
It follows that in the limit $\epsilon \to 0$ the Born expansion for H_α is obtained as a precise result.
Also, the low-energy asymptotic formulae for the distribution of the (negative) eigenvalue are based on the properties of operators with Toepliz (or Toepliz-like) kernel, just as in the proof of Minlos and Faddaev.
There is therefore an evident connection between the two proofs, and "therefore" between the two effects.
And in a sense the result about the distribution of eigenstates which asymptotic for the potential $V(x)$ become exact for point interactions since the operators involved are approximated by Toepliz class.
There is also a major difference, as in the limit $\epsilon \to 0$ in $V_\epsilon(\frac{x}{\epsilon}$ there is a change of scale in energy, with the result that the lower bound of negative spectrum of the three-body hamiltonian is stretched to $-\infty$.
This makes it difficult to see what may be expected from intermediate (physical) situations of forces of "very short " range and of "very large" scattering length, since in the Efimov effect the asymptotic holds for small energies while the approximation by zero-range forces is better for not so small energies (recall the comment we have made on perturbing the system).
Estimates of the errors would be precious but they are very difficult. At present it is unclear if the states of the trimers in Nuclear Physics are part of an "Efimov series.

Remark

An interesting effect, that is related to the Efimov effect but has a simpler proof, is the following [SK] (conspiracy of potential wells).
In a three dimensional system consider the Schroedinger operator

$$H_V = -\Delta - V(x), \quad V = V_1(x) + V_2(x), \quad V_i \geq 0 supp V_1(x) \cap supp V_2(x) = \emptyset$$

Under the assumption that both $-Delta+V_1$ and $-Delta+V_2$ have a zero energy resonance the Hamiltonian H_V has a finite number of bound states.
The proof is obtained also here by using the Birman-Schwinger kernel and estimating the number of eigenvaulues in $(0, 1]$.
The Efimov effect may be thought as a repeated manifestation of this phenomenon, in which the potentials relative to two channels are coupled through the fact that the barycenter is forced to move freely.
♣

Conclusions

We conclude with a more optimistic comment.
We have stated "in passing" that in three body problem with point interactions, if two of the particles are identical fermions and if the mass of third one is sufficiently large, the energy spectrum of the system is bounded below, and there are infinitely many bound states accumulating to the onset of the continuum spectrum.
Under these conditions, preliminary estimates point to the fact that the lower bound of the spectrum for the system obtained by adding a further identical electron is below the energy of the ground state for the three body system.
This and easy compactness estimates implies that there exist at least one bound state of the four body system. The presence of such state has been reported [CMP], but the evidence is still poor.

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