# Completeness and completion: a quick revision.

Alessandro

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Let me just summarise here the remarks I made in the last tutorial. Completeness is an assumption we have been adopting since the very definition of a Hilbert space (a complete inner product space); also $C^*$-algebras are complete as Banach spaces. It is a technical, yet very deep assumption. It usually makes the structure we intend to work with somehow "overabundant" (e.g., a Hilbert space is complete but there is no reason to expect that there is a physical meaning for all (normalised) vectors in a Hilbert space), yet rich enough to accommodate crucial theorems in the theory. Then for many practical purposes we do not need the whole complete space, but only a suitable dense of it. This last fact should not convey the message, though, that completeness is just "decorative".

To start with:

- a sequence $(x_n)_{n=1}^\infty$ in a metric space $(X,d)$ is called a CAUCHY sequence if $\forall\varepsilon>0\ \exists N\in\mathbb{N}$ such that $n,m\geqslant N\Rightarrow d(x_n,x_m)<\varepsilon$;
- if a sequence $(x_n)_{n=1}^\infty$ in a metric space $(X,d)$ converges to some $x\in X$ then $(x_n)_{n=1}^\infty$ is Cauchy; the converse is false in general: usually the first example one encounters is a sequence of rational approximants of $\sqrt{2}$ that is Cauchy in $\mathbb{Q}$ but has no rational limit; analogously, the sequence $(p_n)_{n=1}^\infty$ of polynomials $p_n(x):=\sum_{j=0}^n (j!)^{-1}x^j$ in the metric space consisting of all polynomials on $[0,1]$ with metric $d(p,q):=\sup_{x\in[0,1]}|p(x)-q(x)|$ is Cauchy but it has no limit in the considered space;
- a metric space in which all Cauchy sequences converge in the space is said to be COMPLETE.

The spaces $\mathbb{R}$, $\mathbb{C}$, $C([0,1])$, $L^p(\mathbb{R})$, $1\leqslant p\leqslant\infty$, etc. (with their natural metric structure) are all well-known examples of complete spaces. They are all so much "populated" that there is no alternative for a Cauchy sequence but to converge to a point of the space. This suggests the idea that if a space is not complete one can make it complete by adding a suitable amount of missing points. This is not trivial, because of the constraints imposed by the structure of the space. In fact this procedure is always possible (Completion Theorem), as I will know sketch.

Let $(X,d)$ be a metric space. Let $\mathcal{C}_X$ be the set of Cauchy sequences in $X$. Define a relation $\sim$ in $\mathcal{C}_X$ by declaring "$(s_n)_{n=1}^\infty\sim(t_n)_{n=1}^\infty$" to mean that $d(s_n,t_n)\to 0$ as $n\to\infty$.

(i) The relation "$\sim$" is an equivalence relation.

(ii) Let $\widetilde{X}$ denote the set of equivalence classes of $\mathcal{C}_X$ and let $\widetilde{s}$ denote the equivalence class of $\langle s\rangle\equiv(s_n)_{n=1}^\infty$. The function $\widetilde{d}:\widetilde{X}\times\widetilde{X}\to[0,\infty)$ defined by

$$\widetilde{d}(\widetilde{s},\widetilde{t}\,)\;:=\;\lim_{n\to\infty}d(s_n,t_n)\qquad \widetilde{s},\widetilde{t}\in\widetilde{X}$$

is well-defined (i.e., it does not depend on the choice of the representative in each equivalence class) and is a metric on $\widetilde{X}$.

(iii) The metric space $(\widetilde{X},\widetilde{d})$ is complete.

(iv) For $x\in X$ define $\widetilde{x}$ to be the equivalence class of the constant sequence $\{x,x,x,\dots\}$. The function $x\mapsto\widetilde{x}$ is an isometry of $(X,d)$ onto a dense subset of $(\widetilde{X},\widetilde{d})$.

The above steps yield the following result: any $(X,d)$ is isometrically embedded in a dense of a complete $(\widetilde{X},\widetilde{d})$. In the construction above this is done by identifying each $x\in X$ with the equivalence class $\widetilde{x}$ of the constant sequence $\{x,x,x,\dots\}$.

Such $(\widetilde{X}, \widetilde{d})$, together with the isometric embedding $\phi : X \to \phi(X) \subset \widetilde{X}$, is called a COMPLETION of $(X, d)$.

Something more is true: Assume that $(\widetilde{X}_1, \widetilde{d}_1, \phi_1)$ and $(\widetilde{X}_2, \widetilde{d}_2, \phi_2)$ are two completions of $(X, d)$. Then $\phi_2 \circ \phi_1^{-1}$ maps isometrically $\phi_1(X)$ onto $\phi_2(X)$ and extends to an isometry $\widetilde{X}_1 \to \widetilde{X}_2$. In other words: the completion of a metric space is *unique, up to isometry.*

*Completion of normed spaces.* If $X$ is not only a metric space but also a normed space, then its completion has a natural Banach-space structure.

(i) Let $\widetilde{X}$ be the metric space obtained as the completion of the normed space $X$. $\widetilde{X}$ is naturally equipped with a vector space structure by setting, for all scalars $\lambda, \mu$ and all $x, y \in \widetilde{X}$, $\lambda x + \mu y$ to be the limit in $\widetilde{X}$ of the Cauchy sequence $(\lambda x_n + \mu y_n)_{n=1}^{\infty}$, where $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are sequences in $X$ converging, respectively, to $x$ and to $y$.

(ii) With the above notation, $\|x\| := d(x, 0) = \lim_{n\to\infty} \|x_n\|$ defines a norm on this vector space, turning $\widetilde{X}$ into a Banach space.

The way the Completion Theorem works tells us that taking the completion is a canonical construction, it is sort of "automatic". We are given a non-complete metric or normed space and intrinsically we know and can manipulate its completion. Passing from a vector space to a normed space requires the conceptual step of adding a normed structure compatible with the linear structure; passing from a normed space to a complete normed space (i.e., a Banach space) is an automatism. (One might say that the one who invented normed vector spaces automatically invented also Banach spaces...)

What does completeness "really" mean and how do we use it?

Probably the property of completeness that most reflects the intuitive idea that a complete space is "heavily packed" is Baire's Category Theorem: *a complete metric space cannot be the union of a countable number of sets, the closure of each of which has empty interior.* Thus, if $X$ is complete, one cannot "slice" it as a union $X = \bigcup_{n=1}^{\infty} S_n$ where each $\overline{S_n}$ has empty interior.

A number of pivotal facts in the theory of complete spaces do follow from Baire's Category Theorem: Uniform Boundedness, Open Mapping, Inverse Mapping, Closed Graph, etc.: although in practice we do not care much of the presence of that "abundance" of elements in the space which make it complete, it is the completeness of the space that makes these facts true and applicable. It is very instructive that you revisit the proof of such major theorems and reconstruct how one follows from another and what the crucial role of completeness is – so how they fail in the absence of completeness.

The role of completeness in the theory of Hilbert spaces deserves at least one separate remark.[1] If $\mathcal{H}$ is an inner product space (i.e., a vector space with a scalar product and, therefore, an associated norm), $S$ is a *closed* subspace of $\mathcal{H}$, and $x$ is a point of $\mathcal{H}$, then in general it is false that there exists a unique point $x_S \in S$ of minimum distance from $x$, i.e., a unique point $x_S \in S$ such that $\inf_{s\in S} \|x - s\| = \|x - x_S\|$. This is unfortunate, for one would like very much to have this basic Euclidean geometric property to be true also when the space is infinite-dimensional. Instead, the existence of a unique point in $S$ of minimum distance from $x$ is true *if and only if* $\mathcal{H}$ is complete, namely $\mathcal{H}$ is a Hilbert space (Projection Theorem). One of the major consequences of the Projection Theorem for Hilbert spaces is the Riesz Representation Lemma: the elements of the dual of a Hilbert space $\mathcal{H}$, namely the bounded linear functionals on $\mathcal{H}$, are all maps of the form $\langle y, \cdot\rangle$, $y \in \mathcal{H}$, and the functional norm of each $\langle y, \cdot\rangle$ is the same as $\|y\|$. This too would be false if $\mathcal{H}$ was not complete. Thus, it is completeness that ensures the duality $|\psi\rangle \leftrightarrow \langle\psi|$ between kets and bras!

[1] Have some fun in looking up at this:
`http://mathoverflow.net/questions/35840/the-role-of-completeness-in-hilbert-spaces`