PROBLEMS IN CLASS – Tutorials of 11 and 12 December 2012

Info: www.math.lmu.de/~michel/WS12_MQM.html

Problem 7. Let $\mathcal{H}$ be a separable Hilbert space with norm $\| \ \|$ and scalar product $\langle\ ,\ \rangle$ and let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$. Show that the following two statements are equivalent.

(i) $x_n \rightharpoonup 0$ (i.e., x_n converges weakly to 0) as $n \to \infty$,

(ii) $\langle e_m, x_n\rangle \xrightarrow{n\to\infty} 0$ for each $m \in \mathbb{N}$ and $\sup\limits_{n\in\mathbb{N}} \|x_n\| < C$ for some constant $C > 0$.

Problem 8. (Useful identities and inequalities involving resolvents.) Let $\mathcal{H}$ be a Hilbert space. As usual, given $T \in \mathcal{B}(\mathcal{H})$, $\rho(T)$ is the resolvent set for T and $R_\lambda(T)$, $\lambda \in \rho(T)$, is the resolvent of T at λ, that is, $R_\lambda(T) = (\lambda \mathbb{1} - T)^{-1}$. Prove the following:

(i) $R_\lambda(T) - R_\mu(T) = (\mu - \lambda) R_\lambda(T) R_\mu(T) \quad \forall \lambda, \mu \in \rho(T)$.

(ii) $R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(T - S) R_\lambda(S) \quad \forall \lambda \in \rho(T) \cap \rho(S)$.

Problem 9. Prove that $\displaystyle\int_{-\infty}^{+\infty} \frac{\sin^4 x}{x^4}\,\mathrm{d}x = \frac{2\pi}{3}\,.$

Problem 10. Let d be a positive integer, let $\alpha > 0$, and define

$$H^\alpha(\mathbb{R}^d) \ := \ \Big\{ f \in L^2(\mathbb{R}^d) \mid \|f\|^2_{H^\alpha} := \int_{\mathbb{R}^d} \big(1 + (2\pi|k|)^{2\alpha}\big) |\widehat{f}(k)|^2 \,\mathrm{d}k \ < \ \infty \Big\}.$$

By analogy with the proof of the Sobolev inequality for the space $H^1(\mathbb{R}^2)$ (Theorem 12.3 of the handout "*Crash course in Analysis*"), prove that if $d > 2\alpha$ and $p \in [2, \frac{2d}{d-2\alpha})$ then

$$\|f\|_p \ \leqslant \ C\,\|f\|_{H^\alpha} \qquad \forall f \in H^\alpha(\mathbb{R}^d)$$

for some constant C depending on d, p, and α, but independent of f.