Mathematical Quantum Mechanics

TMP Programme Munich – winter term 2012/2013

PROBLEMS IN CLASS

Some of these problems will be discussed during the tutorials on 4/5 December 2012.

Info: `www.math.lmu.de/~michel/WS12_MQM.html`

Problem 1. Let $f \in L^p(\Omega)$, $1 \le p < \infty$. Prove the dual characterisation of the norm of f,

$$\|f\|_p = \sup_{\substack{g \in L^q(\Omega) \\ \|g\|_q = 1}} \left| \int_\Omega f g \, \mathrm{d}x \right| = \sup_{\substack{g \in \mathcal{D} \\ \|g\|_q = 1}} \left| \int_\Omega f g \, \mathrm{d}x \right|$$

where $\mathcal{D}$ is a dense subspace of $L^q(\Omega)$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Problem 2. Recall the notation

$$|\alpha| := \alpha_1 + \cdots + \alpha_d \,, \qquad D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \,, \qquad x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$$

for a d-dimensional multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Recall also that for any positive integer k,

$$D^k := \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| = k}} D^\alpha$$

(In particular, there are $\frac{d(d+1)}{2}$ terms in $D^2 = \sum_{i \leqslant j} \frac{\partial^2}{\partial x_i \partial x_j}$ and only d terms in $\Delta = \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$).

(i) Prove that $\|\nabla f\|_2^2 \leqslant \|\Delta f\|_2 \|f\|_2 \; \forall f \in \mathcal{S}(\mathbb{R}^d)$.

(ii) Prove that $\|D^2 f\|_2 \leqslant \frac{d(d+1)}{2} \|\Delta f\|_2 \; \forall f \in \mathcal{S}(\mathbb{R}^d)$.

(iii) Prove that for any multi-index α there exists a constant $C_{d,|\alpha|}$ such that

$$\|D^{|\alpha|} f\|_2^2 \;\leqslant\; C_{d,|\alpha|} \, \|\Delta^{|\alpha|} f\|_2 \, \|f\|_2 \qquad \forall f \in \mathcal{S}(\mathbb{R}^d) \,.$$

(iv) Let $d > 2$, $p \in [2, \frac{2d}{d-2}]$, $a := d(\frac{1}{2} - \frac{1}{p})$. Prove that there exists a constant $C_{d,p}$ such that

$$\|f\|_p \;\leqslant\; C_{d,p} \, \|\nabla f\|_2^a \, \|f\|_2^{1-a} \qquad \forall f \in \mathcal{S}(\mathbb{R}^d) \,.$$

(v) Let $d > 2$, $p \in [2, \frac{2d}{d-2}]$, $b := \frac{d}{2}(\frac{1}{2} - \frac{1}{p})$. Deduce from (i) and (iv) that there exists a constant $C_{d,p}$ such that

$$\|f\|_p \;\leqslant\; C_{d,p} \, \|\Delta f\|_2^b \, \|f\|_2^{1-b} \qquad \forall f \in \mathcal{S}(\mathbb{R}^d) \,.$$

Problem 3. Let $d \in \mathbb{N}$, $d \geq 3$, and let $V \in L^{d/2}(\mathbb{R}^d)$ be real-valued. Define the energy functional

$$\mathcal{E}[\psi] := \int_{\mathbb{R}^d} |\nabla \psi(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}^d} V(x)|\psi(x)|^2 \,\mathrm{d}x$$

for $\psi \in H^1(\mathbb{R}^d)$. Prove that for $\|V\|_{d/2}$ sufficiently small, the ground state energy

$$E_0 := \inf\{\mathcal{E}[\psi] \,:\, \psi \in H^1(\mathbb{R}^d), \|\psi\|_2 = 1\}.$$

is non-negative, $E_0 \geq 0$.

Problem 4. Let $\rho \in L^{p_1}(\mathbb{R}^3) \cap L^{p_2}(\mathbb{R}^3)$ with $1 \leqslant p_1 < \frac{3}{2} < p_2 \leqslant \infty$.

(i) Prove that the function $\frac{1}{|\cdot|} * \rho : \mathbb{R}^3 \to \mathbb{C}$ is bounded, continuous, and vanishes as $|x| \to \infty$.

(ii) Using Young's inequality, prove the bound

$$\Big\| \frac{1}{|\cdot|} * \rho \Big\|_\infty \leqslant C_{p_1,p_2} \|\rho\|_{p_1}^{a_1} \|\rho\|_{p_2}^{a_2} \qquad (\bullet)$$

where $a_1 = (\frac{2}{3} - \frac{1}{p_2})/(\frac{1}{p_1} - \frac{1}{p_2})$ and $a_2 = (\frac{1}{p_1} - \frac{2}{3})/(\frac{1}{p_1} - \frac{1}{p_2})$, and where the constant C_{p_1,p_2} depends on p_1 and p_2 only and blows up as $p_1 \to \frac{3}{2}$ or $p_2 \to \frac{3}{2}$.

(iii) Let $p_1 = \frac{3}{2} - \varepsilon$ and $p_2 = \frac{3}{2} + \varepsilon$ with $\varepsilon > 0$. Show that as $\varepsilon \to 0$ the product of the two norms in the R.H.S. of $(\bullet)$ converges to $\|\rho\|_{3/2}$ (while C_{p_1,p_2} obviously blows up), but an inequality of the form $\| |\cdot|^{-1} * \rho\|_\infty \leqslant C\|\rho\|_{3/2}$ is false.

Problem 5. Decide which of the following spaces are C^* algebras.

(i) $\mathbb{C}^n$ with the component-wise product and the norm $\|v\|_p := (\sum_{i=1}^n |v_i|^p)^{1/p}$, $1 \leq p < \infty$, and $\|v\|_\infty = \max_i |v_i|$ for $v = (v_1, \dots, v_n)$.

(ii) $\mathfrak{M}_n(\mathbb{C})$, the set of $n \times n$ matrices with complex entries, with the norm $\|M\| = (\mathrm{tr} M^* M)^{1/2} = \left(\sum_{i,j} |M_{ij}|^2\right)^{1/2}$ for $M = (M_{ij})$.

(iii) $\mathfrak{M}_n(\mathbb{C})$ with the norm $\|M\|^2 = \sup_{v \in \mathbb{C}^n : \|v\|_2 = 1} \sum_i |\sum_k M_{ik} v_k|^2$.

(iv) $\mathfrak{P}(K)$, the set of polynomials on a compact set $K \subset \mathbb{C}^n$, with the norm $\|P\| = \sup_{z \in K} |P(z)|$, $z = (z_1, \dots, z_n)$.

(v) $\mathscr{C}^r(K)$, the set of r times continuously differentiable functions f on $K \subset \mathbb{C}^n$, with norm $\|f\| = \sup_{z \in K} |f(z)|$.

(vi) $L^p(\Omega, \mu)$ with the norm $\|f\|_p = (\int |f|^p \,\mathrm{d}\mu)^{1/p}$ for a positive measure μ on Ω.

Problem 6. Let $\mathfrak{A}$ be a C^* algebra with identity $\mathbb{1}$. Prove the following properties of the spectrum of an operator $A \in \mathfrak{A}$.

(i) $\sigma(A)^n \subset \sigma(A^n)$ where $\sigma(A)^n = \{\lambda^n : \lambda \in \sigma(A)\}$

(ii) $\sigma(\lambda \mathbb{1} - A) = \lambda - \sigma(A)$

(iii) $\sigma(A^*) = \overline{\sigma(A)}$