Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
Prof. T. Ø. Sørensen Ph.D, A. Michelangeli Ph.D

THE X-MAS QUIZ, 21 December 2011

Question 1. T is a compact operator on a Banach space X. Then arbitrarily close to T (in operator norm) you can always find

(a) another compact operator $\widetilde{T}$ that is non injective

(b) a finite rank operator

(c) a self-adjoint operator

Question 2. If the distance (in operator norm) between two bounded operators S, T is less than 1, what is at most the distance between their resolvents (at a point $\lambda \in \rho(T) \cap \rho(S)$)?

(a) $\|R_\lambda(T)\|/\|R_\lambda(S)\|$

(b) $\|R_\lambda(T)\|\,\|R_\lambda(S)\|$

(c) $\|R_\lambda(T)\| + \|R_\lambda(S)\|$

Question 3. T is a bounded linear map on $L^2[0,1]$ that is invertible with bounded inverse. What is true then?

(a) T cannot be self-adjoint

(b) T cannot be unitary

(c) T cannot be compact

Question 4. A bounded linear map $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is such that the region $\{z \in \mathbb{C} \,|\, \mathrm{i}z \in [1,3]\}$ consists of eigevalues of S. Which among the statements below is false?

(a) S is not compact

(b) S is not self-adjoint

(c) $\|S\| \leqslant 3$

Question 5. One can say that the whole construction of the continuos functional calculus $f(A)$ of a self-adjoint operator A consists of "standard arguments" plus one "crucial step". Which one and in which sense?

(a) restrict the construction to $p(A)$, where p is just a polynomial

(b) prove the uniqueness of the functional calculus $p \mapsto p(A)$ when $p(t) = 1$ and $p(t) = t$

(c) prove that $\sigma(p(A)) = p(\sigma(A))$ when p is a polynomial

(d) prove that $\|p(A)\| = \sup_{\lambda \in \sigma(A)} |p(\lambda)|$ when p is a polynomial

Question 6. What is *false* about the space $\mathcal{M}_B(K)$ of bounded, Borel-measurable, complex-valued functions on the compact subset K of $\mathbb{C}$?

(a) $\mathcal{M}_B(K)$ is the smallest span of functions $K \to \mathbb{C}$ containing the continuous functions and closed under pointwise limits of uniformly bounded sequences

(b) $\mathcal{M}_B(K)$ is the clousure, with respect of the uniform norm, of the space of step functions $K \to \mathbb{C}$

(c) $\mathcal{M}_B(K)$ is the clousure, with respect of the uniform norm, of the space of step functions $K \to \mathbb{C}$ built on Borel sets of K

Question 7. What is the "most relevant" class of non-continuous functions to which one wants to extend the functional calculus and why?

(a) the functions with an integrable local singularity

(b) the characteristic functions

(c) functions of the form $f * g$ where $f \in C_0^\infty(\mathbb{R})$ and g is bounded measurable

Question 8. A is a self-adjoint operator with spectrum $\sigma(A) = [0,1]$. For which of the following functions $f : [0,1] \to \mathbb{R}$ it is possible to find a polynomial p such that the operator $f(A)$ is approximated with the desired precision in operator norm by the operator $p(A)$?

(a) $f(x) = e^{-x^2} \arctan(x\sqrt{2})$

(b) $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} \cos(2012^n \pi x)$

(c) $f(x) = x^2 \mathrm{sgn}(x - \frac{1}{2})$

Question 9. Which of the following conditions on a bounded operator T between Banach spaces guarantees that the range of T is a closed set?

(a) $\|Tx\| < \|x\| \ \forall x$

(b) $\|Tx\| = \|x\| \ \forall x$

(c) $\|Tx\| > \|x\| \ \forall x$

(d) $0 \in \rho(T)$

(e) $0 \in \sigma(T)$