Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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PROBLEM IN CLASS – WEEK 6

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/WS11-12_FA2.html`.

Problem 21. (Positive bounded operators on a complex Hilbert space are self-adjoint)

Let $\mathcal{H}$ be a *complex* Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$ be such that $\langle x, Ax\rangle \geqslant 0$ for all $x \in \mathcal{H}$. Show that $A = A^*$. (*Hint:* polarise.)

Problem 22. Recall from class that the square root of a positive compact operator T on a Hilbert space $\mathcal{H}$ is straightforwardly constructed via the Spectral Theorem for compact operators in the following way: if $T = \sum_n \lambda_n |e_n\rangle\langle e_n|$ is the spectral decomposition of T, where each $\lambda_n \geqslant 0$ and $\{e_n\}_n$ is an orthonormal system in $\mathcal{H}$, then $\sqrt{T} := \sum_n \sqrt{\lambda_n}|e_n\rangle\langle e_n|$. The construction of $\sqrt{T}$ for generic $T \in \mathcal{B}(\mathcal{H})$, $T \geqslant \mathbb{O}$ (not necessarily compact) is still possible but is somehow tedious if one proceeds from scratch, as you are asked to do in this problem (of course the Functional Calculus (Spectral Theorem) for bounded self-adjoint operator would do the job in one shot).

(i) Show that the power series

$$\sqrt{1-x} \;=\; \sum_{n=0}^{\infty} c_n x^n \;=\; 1 - \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 - \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 - \cdots$$

converges absolutely for $|x| \leqslant 1$.

(ii) Let $T \in \mathcal{B}(\mathcal{H})$, $T \geqslant \mathbb{O}$. Show that

$$S \;:=\; \sqrt{\|T\|}\,\sum_{n=0}^{\infty} c_n \Big(\mathbb{1} - \frac{T}{\|T\|}\Big)^n$$

is well defined as a series that converges in operator norm, $S \geqslant 0$, and $S^2 = T$.

(iii) Show that the operator $S \in \mathcal{B}(\mathcal{H})$ such that $S \geqslant 0$ and $S^2 = T$ is unique.

Problem 23.

(i) Let $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & -2 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Show that it is false that $|A+B| \leqslant |A| + |B|$.

(ii) Show that $|A+B|^2 \leqslant 2(|A|^2 + 2|B|^2)$ for every $A, B \in \mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ is a Hilbert space).

Problem 24. Let $\mathcal{H}$ be a Hilbert space and $U \in \mathcal{B}(\mathcal{H})$. (More generally the following applies to $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two possibly distinct Hilbert spaces.) Recall the definitions:

- U is a unitary operator if U is surjective and $\|Ux\| = \|x\|$ $\forall x \in \mathcal{H}$.
- U is an isometry if $\|Ux\| = \|x\|$ $\forall x \in \mathcal{H}$.
- U is a partial isometry if $\|Ux\| = \|x\|$ $\forall x \in (\operatorname{Ker} U)^{\perp}$.
  $(\operatorname{Ker} U)^{\perp}$ is called the initial space of U and $\operatorname{Ran} U$ is called the final space of U.

Prove the following.

(i) U is unitary $\Leftrightarrow$ $U^*U = UU^* = \mathbb{1}$.

(ii) U is an isometry $\Leftrightarrow$ $U^*U = \mathbb{1}$.

(iii) If U is a partial isometry then $\operatorname{Ran} U$ is closed in $\mathcal{H}$ and if $U \neq \mathbb{O}$ then $\|U\| = 1$.

(iv) The adjoint of a partial isometry is a partial isometry with initial space and final space interchanged.

(v) U is a partial isometry $\Leftrightarrow$ U^*U is an orthogonal projection, in which case
  $\rightarrow$ U^*U is the orthogonal projection onto the initial space of U,
  $\rightarrow$ UU^* is the orthogonal projection onto the final space of U.

(vi) U is a partial isometry $\Leftrightarrow$ $U = UU^*U$.