Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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PROBLEM IN CLASS – WEEK 4

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/WS11-12_FA2.html`.

Problem 13. (Spectrum of the product.) Let X be a Banach space and $T, S \in \mathcal{B}(X)$.

(i) Show that $\sigma(TS) \cup \{0\} = \sigma(ST) \cup \{0\}$ (i.e., the non-zero elements of $\sigma(TS)$ and $\sigma(ST)$ are the same).

(ii) Show that $\sigma(TS) = \sigma(ST)$ need not hold.

Problem 14. (When the space is reflexive an operator is compact iff it is "vollstetig").
Let X be a reflexive Banach space, that is, a Banach space such that $X \cong X^{**}$ via the canonical isomorphism $x \stackrel{\cong}{\longmapsto} \langle x, \cdot \rangle_{X^{**}, X^*}$ where $\langle x, \cdot \rangle_{X^{**}, X^*} \in X^{**}$ acts on every $\phi \in X^*$ via the duality $\langle x, \phi \rangle_{X^{**}, X^*} = \phi(x)$ (and recall also that $\| \langle x, \cdot \rangle_{X^{**}, X^*} \|_{X^{**}} = \|x\|_X$). Let $T \in \mathcal{B}(X)$. Show that

$$T \text{ is compact} \qquad \Leftrightarrow \qquad \begin{cases} \text{for every sequence } \{x_n\}_{n=1}^\infty \text{ in } X, \\ x_n \xrightarrow[n\to\infty]{\text{weak}} x \;\; \Rightarrow \;\; Tx_n \xrightarrow[n\to\infty]{\|\,\|} Tx \,. \end{cases}$$

(*Hint:* Banach-Alaoglu.)

Problem 15. Let $k \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ for some positive integer d. Use the notation $k(x, y)$ with $x, y \in \mathbb{R}^d$. Consider the bounded linear operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ defined by

$$(Tf)(x) := \int_{\mathbb{R}^d} k(x, y) f(y) \mathrm{d}y \qquad \text{for a.e. } x \in \mathbb{R}^d \,.$$

(The fact that $T \in \mathcal{B}(L^2(\mathbb{R}^d))$ was proved in Problem 9.) Show that

$$\dim \operatorname{Ker}(\mathbb{1} - T) \;\leqslant\; \|k\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \,.$$

Problem 16. Let X be a Banach space, $T \in \mathcal{B}(X)$, and p a polynomial with complex coefficients and degree $\geqslant 1$. Show that

$$\sigma(p(T)) \;=\; p(\sigma(T)) \;:=\; \{\, p(\lambda) \in \mathbb{C} \,|\, \lambda \in \sigma(T) \,\} \,.$$