Institute of Mathematics, LMU Munich – Winter Term 2011/2012
Prof. T. Ø. Sørensen Ph.D, A. Michelangeli Ph.D

HOMEWORK ASSIGNMENT no. 10, issued on Wednesday 21 December 2011
Due: Wednesday 11 January 2012 by 2 pm in the designated "FA2" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS11-12_FA2.html`

Each exercise sheet is worth a full mark of 40 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 37. Let $\mathcal{H}$ be a Hilbert space and let A, B be bounded self-adjoint operators on $\mathcal{H}$.

(i) Assume that $A \leqslant B$. Show that $C^*AC \leqslant C^*BC$ for all $C \in \mathcal{B}(\mathcal{H})$.

(ii) Assume that $\mathbb{O} \leqslant A \leqslant B$. Show that $\|A\| \leqslant \|B\|$.

(iii) Assume that $A \geqslant \mathbb{O}$. Show that A is invertible if and only if $A \geqslant c\mathbb{1}$ for some $c > 0$.

(iv) Assume that $\mathbb{O} \leqslant A \leqslant B$. Show that for every $\lambda > 0$ $A + \lambda\mathbb{1}$ and $B + \lambda\mathbb{1}$ are positive and invertible and $(B + \lambda\mathbb{1})^{-1} \leqslant (A + \lambda\mathbb{1})^{-1}$

(v) Assume that $\mathbb{O} \leqslant A \leqslant B$ and that A is invertible. Show that B is invertible too and $B^{-1} \leqslant A^{-1}$. (*Hint:* (iii) and (iv) above.)

Exercise 38. (Absolute value, positive, negative part of a self-adjoint operator: with and without the functional calculus.)

Let $\mathcal{H}$ be a Hilbert space and let $A = A^* \in \mathcal{B}(\mathcal{H})$.

(i) Explain why the operator $|A| = \sqrt{A^*A}$ constructed with Hilbert space techniques (see, e.g., Problem 22) and the operator $|A|$ constructed by means of the continuous functional calculus are actually the same.

(ii) Show that the limit, in operator norm, of a sequence of positive operators on $\mathcal{H}$ is positive.

(iii) Show that $A_n := 2(\frac{4}{n}\mathbb{1} + (|A| - A)^2)^{-1}(|A| - A)^2|A|$ is bounded and positive in $\mathcal{B}(\mathcal{H})$ for every $n \in \mathbb{N}$. (*Hint:* the operators A, $|A|$, $(\frac{4}{n}\mathbb{1} + (|A| - A)^2)^{-1}$ commute and the latter is positive, then use the same argument as in Exercise 37 (i). No functional calculus argument is needed here, although it would help.)

(iv) Show that $A_n \xrightarrow[n\to\infty]{\|\,\|} |A| - A$ and deduce by (ii) that $A \leqslant |A|$.

(v) Re-prove that $A \leqslant |A|$ using the continuous functional calculus.

(vi) Show that there is a unique pair of positive operators A_+, A_- in $\mathcal{B}(\mathcal{H})$ such that $A_+A_- = \mathbb{O}$ and $A = A_+ - A_-$. (*Hint:* both to prove that $A_+ \geqslant 0$, $A_- \geqslant 0$, and to prove uniqueness, you need $A \leqslant |A|$.)

Exercise 39. (Operator monotone functions.)

A continuous, real-valued function f on an interval I is said OPERATOR MONOTONE (on the interval I) if $A \leqslant B \Rightarrow f(A) \leqslant f(B)$ for every bounded, self-adjoint operators A, B on a Hilbert space $\mathcal{H}$ such that $\sigma(A) \subset I$, $\sigma(B) \subset I$.

(i) Show that the function $f_\alpha(t) = \dfrac{t}{(1+\alpha\, t)}$ is operator monotone on $\mathbb{R}^+$ if $\alpha \geqslant 0$.

(ii) Show that the function $f_\alpha(t)$ considered in (i) is operator monotone on $[0,1]$ if $\alpha \in (-1, 0]$.

(iii) Let A, B be bounded self-adjoint operators on a Hilbert space $\mathcal{H}$ such that $\mathbb{O} \leqslant A \leqslant B$. Show that $\mathbb{O} \leqslant \sqrt{A} \leqslant \sqrt{B}$, in other words, $x \mapsto \sqrt{x}$ is operator monotone on $\mathbb{R}^+$.

(*Hint:* Exercise 37 (iv) and the identity $\sqrt{x} = \dfrac{1}{\pi} \displaystyle\int_0^{+\infty} \frac{\mathrm{d}\lambda}{\sqrt{\lambda}} \Big(1 - \frac{\lambda}{\lambda + x}\Big)$, valid $\forall x \geqslant 0$.)

(iv) Same assumption as in (iii). Show that $\mathbb{O} \leqslant A^\alpha \leqslant B^\alpha$ $\forall \alpha \in [0,1]$. (*Hint:* same strategy as in (iii), use now $x^\alpha = \dfrac{\sin \alpha\pi}{\pi} \displaystyle\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^{1-\alpha}} \frac{x}{\lambda + x}$ valid $\forall x \geqslant 0$, $\forall \alpha \in (0,1)$.)

(v) Produce a counterexample to the conclusion in (iv) when $\alpha > 1$.

Exercise 40. (Functional calculus at work.)

(i) Let A be a bounded, self-adjoint, and positive operator on $L^2[0,1]$ such that

$$(A^{2012} e^A) f(x) \;=\; e f(x) + e \int_0^x f(y)\,\mathrm{d}y \qquad \forall f \in L^2[0,1] \text{ and a.e. } x \in [0,1]\,.$$

Find $\sigma(A)$. (*Hint:* spectral mapping theorem.)

(ii) Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$ such that $\mathbb{O} \leqslant A \leqslant \mathbb{1}$. Find a sequence $\{P_n\}_{n=1}^\infty$ of pairwise commuting orthogonal projections on $\mathcal{H}$ such that

$$A \;=\; \sum_{n=1}^\infty 2^{-n} P_n\,.$$

(*Hint:* reconstruct the function $f(x) = x$ as a sum of step functions.)

(iii) For every $\varepsilon > 0$ consider the function G defined on $x \in \mathbb{R}$ by

$$G_\varepsilon(x) \;:=\; \frac{1}{\pi \mathrm{i}} \int_0^1 \Big(\frac{1}{x - (t + \mathrm{i}\varepsilon)} - \frac{1}{x - (t - \mathrm{i}\varepsilon)} \Big)\,\mathrm{d}t\,.$$

Consider also the operator A of part (i). Show that $G_\varepsilon(A)$ is a well-defined bounded operator on $L^2[0,1]$ and that

$$G_\varepsilon(A) f \xrightarrow[\varepsilon \to 0]{\|\,\|_2} f \qquad \forall f \in L^2[0,1]\,.$$

Christmas puzzle. (Not to be marked.) Determine all operators A and B in $\mathcal{B}(\mathcal{H})$ ($\mathcal{H}$ being a Hilbert space) such that B is invertible and $A^n \to B$ as $n \to \infty$.