Functional Analysis II

Institute of Mathematics, LMU Munich – Winter Term 2011/2012
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HOMEWORK ASSIGNMENT no. 6, issued on Wednesday 23 November 2011
Due: Wednesday 30 November 2011 by 2 pm in the designated "FA2" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS11-12_FA2.html`

Each exercise sheet is worth a full mark of 40 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 21. (The Hardy operator – I)

Consider the Banach space $C([0,1],\mathbb{R})$ of *real*-valued continuous functions on $[0,1]$ equipped with the usual norm of the maximum $\|f\| = \max_{x\in[0,1]} |f(x)|$. Let $f \mapsto Tf$ be the map defined by

$$(Tf)(x) \;:=\; \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x f(y)\,\mathrm{d}y & \text{if } x \in (0,1] \\ \\ f(0) & \text{if } x = 0\,. \end{cases} \qquad (*)$$

(i) Show that $(*)$ defines a linear map T of $C([0,1],\mathbb{R})$ into itself.

(ii) Show that T is bounded and compute $\|T\|$.

(iii) Show that $\sigma_{\mathrm{p}}(T) = (0,1]$ and for each eigenvalue determine the corresponding eigenfunctions.

(iv) Decide whether T is compact or not.

Exercise 22. (The Hardy operator – II)

Next to $C([0,1],\mathbb{R})$ considered in Exercise 21, consider the Banach space $C^1([0,1],\mathbb{R})$ equipped with norm $\|f\|' = \max_{x\in[0,1]} |f(x)| + \max_{x\in[0,1]} |f'(x)|$ and the Banach space $L^q[0,1]$ with $1 \leqslant q < \infty$. Let T be the map defined in $(*)$, Exercise 21.

(i) Show that $(*)$ defines a bounded linear map T of $C^1([0,1],\mathbb{R})$ into itself.

(ii) Show that $\sigma_{\mathrm{p}}(T) = (0,\frac{1}{2}] \cup \{1\}$ and for each eigenvalue determine the corresponding eigenfunctions.

(iii) Show that $(*)$ defines a bounded linear map $T : C([0,1],\mathbb{R}) \to L^q[0,1]$ and decide whether such a map is compact or not.

Exercise 23. (The Hardy operator – III)

Let $p \in (1,\infty)$. For every $f \in L^p[0,1]$ consider the map $f \mapsto Tf$ defined by

$$(Tf)(x) \;:=\; \frac{1}{x}\int_0^x f(y)\,\mathrm{d}y \qquad \text{for almost every } x \in [0,1]\,.$$

(i) Show that T defines a bounded linear map $T : L^p[0,1] \to L^p[0,1]$.

(*Hint:* Find first a bound $\|Tf\|_p \leqslant C\|f\|_p$ for suitable smooth f's vanishing at $x=0$, by means of a convenient integration by parts in $\int_0^1 |(Tf)(x)|^p \mathrm{d}x$. Then complete the argument by density.)

(ii) Compute $\|T\|$.

(*Hint:* you may check that the interval $(0,\frac{p}{p-1})$ is all made of eigenvalues.)

(iii) Decide whether T is compact or not.

(iv) Decide whether T is compact as a map $T : L^p[0,1] \to L^q[0,1]$ with $1 \leqslant q < p < \infty$.

Exercise 24. (The Hardy operator – IV)

Let $p \in (1,\infty)$. For every $f \in L^p(\mathbb{R}^+)$ consider the map $f \mapsto Tf$ defined by

$$(Tf)(x) \;:=\; \frac{1}{x}\int_0^x f(y)\,\mathrm{d}y \qquad \text{for almost every } x \in \mathbb{R}^+\,.$$

(i) Show that T defines a bounded linear map $T : L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$.

(*Hint:* As in Exercise 23 (i).)

(ii) Deduce from (i) the following inequality $\forall F \in C^1((0,\infty))$ with $F' \in L^p[0,\infty]$, $F(0)=0$:

$$\int_0^\infty \frac{|F(x)|^p}{x^p}\,\mathrm{d}x \;\leqslant\; \Big(\frac{p}{p-1}\Big)^p \int_0^\infty |F'(x)|^p\,\mathrm{d}x$$

(the Hardy's inequality, G. H. Hardy, *Note on a theorem of Hilbert*, Math. Zeitschr. 6 (1920), 314-317).

(iii) Compute $\|T\|$.

(*Hint:* saturate your bound $\|Tf\|_p \leqslant C\|f\|_p$ with a sequence $f_n(x) \sim x^{-\alpha(n)}$ as $x \to \infty$, adjusting the power-law decay $\alpha(n)$ in a convenient way.)

(iv) Decide whether T is compact or not.

(v) Determine the adjoint T' of T.