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HOMEWORK ASSIGNMENT no. 5, issued on Wednesday 16 November 2011
Due: Wednesday 23 November 2011 by 2 pm in the designated "FA2" box on the 1st floor
Info: www.math.lmu.de/~michel/WS11-12_FA2.html

Each exercise sheet is worth a full mark of 40 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 17.

(i) Let X, Y, and Z be Banach spaces with norms $\|\ \|_X$, $\|\ \|_Y$, and $\|\ \|_Z$, respectively. Assume that $X \subset Y$ with compact injection and $Y \subset Z$ with continuous injection. (I.e., $\mathrm{id} : X \to Y$ is compact and $\mathrm{id} : Y \to Z$ is bounded.) Prove that

$$\forall \varepsilon > 0 \ \ \exists C_\varepsilon \geqslant 0 \ \text{ such that } \ \|x\|_Y \leqslant \varepsilon \, \|x\|_X + C_\varepsilon \|x\|_Z \quad \text{for all } x \in X.$$

(ii) Show that $\forall \varepsilon > 0 \ \ \exists C_\varepsilon \geqslant 0$ such that

$$\max_{x\in[0,1]} |f(x)| \ \leqslant \ \varepsilon \max_{x\in[0,1]} |f'(x)| + C_\varepsilon \, \|f\|_{L^1[0,1]} \qquad \forall\, f \in C^1([0,1])\,.$$

Exercise 18. Given a Banach space X, $T \in \mathcal{B}(X)$, and $\lambda \in \mathbb{C}$, a sequence $\{x_n\}_{n=1}^\infty$ in X such that $\|x_n\| = 1 \ \forall n \in \mathbb{N}$ and $\|Tx_n - \lambda x_n\| \xrightarrow{n\to\infty} 0$ is called a Weyl sequence for T at λ.

(i) Let X be a Banach space and let $T \in \mathcal{B}(X)$. Prove the implication

$$T \text{ has a Weyl sequence at } \lambda \in \mathbb{C} \quad \Rightarrow \quad \lambda \in \sigma(T)\,.$$

(ii) Let X be a Banach space and let $T \in \mathcal{B}(X)$. Prove the implication

$$T \text{ has a Weyl sequence at } \lambda \in \mathbb{C} \quad \Leftarrow \quad \lambda \in \partial\sigma(T), \text{ the boundary of } \sigma(T)\,.$$

(iii) Let $\mathcal{H}$ be a Hilbert space and let T be a self-adjoint or a unitary operator in $\mathcal{B}(\mathcal{H})$. Prove the implications

$$T \text{ has a Weyl sequence at } \lambda \in \mathbb{C} \quad \Leftrightarrow \quad \lambda \in \sigma(T)\,.$$

(iv) Let K be a non-empty, compact subset of $\mathbb{C}$. Show that K is the spectrum of a normal bounded operator on a Hilbert space, that is, there is an operator $T \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, such that $\sigma(T) = K$.

Exercise 19.

(i) Produce an example of a compact operator T, other than the Volterra operator discussed in Exercise 10, which does not have eigenvalues, and for which therefore $\sigma(T) = \{0\}$.

(ii) Produce an example of an operator T on an infinite-dimensional Banach space, other than the identity, such that $\|T\| = 1$ and $\sigma(T) = \{1\}$.

Note: there is no room for such an example in the finite-dimensional case! Every finite matrix is unitarily equivalent to a triangular matrix. If a triangular matrix has only 1's on the main diagonal, then its norm is at least 1; the norm can be equal to 1 only if the matrix is the identity. Thus, on a finite-dimensional space the identity is the only operator satisfying the above conditions.

Exercise 20. Let X be a Banach space and let $T \in \mathcal{B}(X)$.

(i) Assume that there is an integer $n \geqslant 2$ such that $T^n = \mathbb{O}$. Prove that $\sigma(T) = \{0\}$ and determine $(\lambda \mathbb{1} - T)^{-1}$ for $\lambda \neq 0$.

(ii) Assume that there is an integer $n \geqslant 2$ such that $T^n = \mathbb{1}$. Prove that $\sigma(T) \subset \{\lambda \in \mathbb{C} \,|\, \lambda^n = 1\}$ and determine $(\lambda \mathbb{1} - T)^{-1}$ for $\lambda^n \neq 1$.

(iii) Assume that there is an integer $n \geqslant 2$ such that $\|T^n\| < 1$. Prove that $\mathbb{1} - T$ is a bijection on X and give an expression of $(\mathbb{1} - T)^{-1}$ in terms of $(\mathbb{1} - T^n)^{-1}$ and the powers of T.