EXERCISE SHEET 12, issued on Tuesday 1 February 2011
Due: Tuesday 8 February 2011 by 8,15 a.m. in the designated "MQM" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS10_MQM.html`

Each exercise is worth a full mark as specified below. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English. The exercise marked with ★ is for extra credit.

Exercise 45 [13 points]. Consider the Hamiltonian of the Helium atom

$$H^{\mathrm{He}} \;=\; -\Delta_{x_1} - \Delta_{x_2} - \frac{2}{|x_1|} - \frac{2}{|x_2|} + \frac{1}{|x_1 - x_2|}$$

acting on the full $L^2(\mathbb{R}^3_{x_1}\times\mathbb{R}^3_{x_2}, \mathrm{d}x_1\,\mathrm{d}x_2)$, i.e., no symmetry constraint is imposed, for simplicity. Recall from Exercise 19 that the ground state energy E_0^{He} satisfies the bounds $-2 \leqslant E_0^{\mathrm{He}} \leqslant -1.42$, whereas experimentally $E_0^{\mathrm{He}} \approx -1.45$. Prove that H^{He} has infinitely many eigenvalues in the interval $[E_0^{\mathrm{He}}, -1)$. (*Hint:* Use the min-max principle and construct for any positive integer N an N-dimensional subspace $\mathcal{H}_N$ of $L^2(\mathbb{R}^3_{x_1}\times\mathbb{R}^3_{x_2}, \mathrm{d}x_1\,\mathrm{d}x_2)$ with $\sup\limits_{\substack{\psi\in\mathcal{H}_N\\ \|\psi\|_2=1}} \langle\psi, H^{\mathrm{He}}\psi\rangle < -1$.)

Exercise 46 [15 points]. Consider the Hamiltonian $H = -\Delta + V$ on $L^2(\mathbb{R}^3)$.

(i) Assume that $V \in L^{3/2}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ and that $V(x) \leqslant -\dfrac{a}{|x|^{2-\varepsilon}}$ whenever $|x| \geqslant R_0$ for some $R_0, a, \varepsilon > 0$. Prove that H has infinitely many *negative* eigenvalues. (*Hint:* as in Exercise 45.)

(ii) Assume that for some $R_0 > 0$ and $b \in [0,1)$ one has $V(x) = -\dfrac{b}{4|x|^2} - W(x)$, where W is a bounded potential such that $W \geqslant 0$ and $\mathrm{supp}(W) \subseteq \{x : |x| \leqslant R_0\}$. Prove that the number of the eigenvalues of H is finite. (*Hint:* split the kinetic energy $-\Delta = -b\Delta - (1-b)\Delta$ and use Hardy's inequality, the Cwikel-Lieb-Rosenbljum bound, and the min-max principle.)

★ **Exercise 47 [15 points].** Consider the Hamiltonian $H = -\Delta - V$ on $L^2(\mathbb{R}^3)$ with $V \in C_0^\infty(\mathbb{R}^3)$ and $V \geqslant 0$. Denote by $E_1 \leqslant E_2 \leqslant \cdots \leqslant 0$ the non-positive eigenvalues of H, listed in increasing order. Let N be their number. For any $E < 0$ denote by $N(E)$ the number of eigenvalues of H strictly smaller than E.

(i) Denote by $\mu_1(\lambda) \leqslant \mu_2(\lambda) \leqslant \cdots \leqslant 0$ the non-positive eigenvalues of $-\Delta - \lambda V$, $\lambda \in (0,1)$, listed in increasing order. Prove that

$$N(E) \;=\; \#\{\, \mu_n \,|\, \mu_n(\lambda) = E \text{ for some } \lambda \in (0,1)\} \;\leqslant\; \sum_{\substack{\lambda \text{ s.t.} \\ \mu_k(\lambda)=E \\ k=1,2,\dots}} \frac{1}{\lambda^2}\,.$$

(ii) Prove that

$$N \;\leqslant\; \frac{1}{(4\pi)^2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{d}x\,\mathrm{d}y\, \frac{V(x)\,V(y)}{|x-y|^2} \;<\; \infty\,.$$

(*Hint:* use the Birman-Schwinger principle to show that λ^{-1} is an eigenvalue of the Birman-Schwinger kernel operator K_E if and only if $\mu_k(\lambda) = E$ and compute $\mathrm{Tr} K_E^* K_E$.)

Exercise 48 [12 points]. The purpose of this exercise is to test how good the Lieb-Thirring inequality is. To this aim you will need the following explicit constants in the Lieb-Thirring inequality:

$$L_{\gamma=1,d=1} \;=\; \frac{4}{3}\,, \qquad L_{\gamma=2,d=1} \;=\; \frac{16}{5\pi}\,.$$

Consider the Hamiltonian $H_M = -\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 - 2M - 1$ acting on $L^2(\mathbb{R})$, where M is a positive integer. Denote by $E_0 \leqslant E_1 \leqslant \cdots \leqslant 0$ the non-positive eigenvalues of H_M, listed in increasing order. For any $\gamma \geqslant \frac{1}{2}$ let $\Sigma_\gamma(M) := \sum_{j\geqslant 0} |E_j|^\gamma$ and let $\Sigma_\gamma^{\mathrm{LT}}(M)$ be the upper bound on $\Sigma_\gamma(M)$ given by the Lieb-Thirring inequality.

(i) Compute $\Sigma_1(M)$ and $\Sigma_2(M)$.

(ii) Compute $\Sigma_1^{\mathrm{LT}}(M)$ and $\Sigma_2^{\mathrm{LT}}(M)$.

(iii) Compute $\displaystyle\lim_{M\to+\infty} \frac{\Sigma_1(M)}{\Sigma_1^{\mathrm{LT}}(M)}$ and $\displaystyle\lim_{M\to+\infty} \frac{\Sigma_2(M)}{\Sigma_2^{\mathrm{LT}}(M)}$.