EXERCISE SHEET 11, issued on Tuesday 25 January 2011
Due: Tuesday 1 February 2011 by 8,15 a.m. in the designated "MQM" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS10_MQM.html`

> *Each exercise is worth a full mark as specified below. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English. The exercise marked with ★ is for extra credit.*

Exercise 41 [12 points]. In part (i) and (iii) of this exercise $^\vee$ denotes the inverse Fourier transform in L^2. In part (ii) $^\vee$ denotes the inverse Fourier transform in L^1. $\widehat{\psi}$ denote the Fourier transform of an L^2-function.

(i) Let $f,g\in L^\infty(\mathbb{R}^d)$. For every $\psi\in L^2(\mathbb{R}^d)$ define

$$\big(f(x)g(-i\nabla)\psi\big)(x)\ :=\ f(x)\,(g(2\pi\cdot)\,\widehat{\psi})^\vee(x)\,. \tag{$*$}$$

Prove that $(*)$ defines an element of $L^2(\mathbb{R}^d)$ and that the map $\psi\overset{T}{\longmapsto} f(x)g(-i\nabla)\psi$ is a bounded operator on $L^2(\mathbb{R}^d)$ with $\|T\|\leqslant\|f\|_\infty\|g\|_\infty$.

(ii) Let $f,g\in L^2(\mathbb{R}^d)$. Prove that $(*)$ defines a Hilbert-Schmidt map $\psi\overset{T}{\longmapsto} f(x)g(-i\nabla)\psi$ with $\|T\|_{\mathrm{HS}}=(2\pi)^{-d/2}\|f\|_2\,\|g\|_2$.

(iii) Let $f,g\in\overline{L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)}^{\|\ \|_\infty}$ (the closure in the L^∞-norm). This is the case, for instance, when $f,g\in L^\infty(\mathbb{R}^d)$ and $f(x),g(x)\to 0$ as $|x|\to\infty$. Prove that $(*)$ defines a compact operator $\psi\overset{T}{\longmapsto} f(x)g(-i\nabla)\psi$.

Exercise 42 [13 points]. Let $E_0^f(N)$ be the ground state energy of N non-interacting fermions with equal masses in d dimensions, under all assumptions of Theorem 11.1 of the handout "*Many-body quantum systems*". In particular, the underlying one-body Hamiltonian $h=-\Delta+V$ is bounded below and admits at least N negative eigenvalues. Show that

$$E_0^f(N)\ =\ \inf\left\{\begin{array}{c}\mathrm{Tr}\,h\gamma\ |\ 0\leqslant\gamma\leqslant\mathbb{1}\,,\ \mathrm{Tr}\,\gamma=N\,,\\ \gamma=\sum_j\mu_j\,|\psi_j\rangle\langle\psi_j|\ \text{(the spectral decomposition of }\gamma)\\ \text{where }\{\psi_j\}_j\text{ is an orthonormal system in }L^2,\\ \psi_j\in H^1(\mathbb{R}^d)\,\forall j\,,\ \sum_j\mu_j\|\nabla\psi_j\|_2^2<\infty\end{array}\right\}.$$

(*Hint:* Theorems 11.1, 8.4, and 8.5 for a lower bound to $E_0^f(N)$, Theorems 9.2, 8.4, 8.5, and 8.3 for an upper bound.)

★**Exercise 43 [15 points].** Consider $g \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^+)$, spherically symmetric and with $\int_{\mathbb{R}^3} g^2 = 1$. For any $\theta > 0$ and $x \in \mathbb{R}^3$ define $g_\theta(x) := \theta^{-3/2} g(x/\theta)$. For any $(q,k) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $x \in \mathbb{R}^3$ define $\psi_{q,k} := e^{2\pi i k \cdot x}\, g_\theta(x-q)$. For any measurable function $a : \mathbb{R}^3 \times \mathbb{R}^3 \to [0,1]$ define the bounded operator $\mathfrak{Z}(a) : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ by

$$\langle \varphi, \mathfrak{Z}(a)\, \psi \rangle \;:=\; \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathrm{d}q\, \mathrm{d}k\; a(q,k)\, \langle \varphi, \psi_{q,k} \rangle\, \langle \psi_{q,k}, \psi \rangle \qquad\qquad \varphi, \psi \in L^2(\mathbb{R}^3)\,.$$

(i) Show that $0 \leqslant \mathfrak{Z}(a) \leqslant \mathbb{1}$.

(ii) Show that $\mathrm{Tr}\,\mathfrak{Z}(a) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathrm{d}q\, \mathrm{d}k\; a(q,k)$.

(iii) Show that $\mathrm{Tr}(-\Delta)\mathfrak{Z}(a) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathrm{d}q\, \mathrm{d}k\; a(q,k) \|\nabla \psi_{q,k}\|_2^2$.

(iv) Show that $\|\nabla \psi_{q,k}\|_2^2 = |2\pi k|^2 + \theta^{-2} \|\nabla g\|_2^2$.

(v) Let $V \in L^\infty(\mathbb{R}^3)$. Show that $\int_{\mathbb{R}^3} \mathrm{d}x\, V(x) |\psi_{q,k}(x)|^2 = (V * g_\theta^2)(q)$.

Exercise 44 [15 points]. Same notation as in Exercise 43 (you may use the results from there without proof.) Consider the potential V such that $V \in L^\infty(\mathbb{R}^3)$ and $V_- \in L^{5/2}(\mathbb{R}^3)$ (V_- is the negative part of V, as usual). Assume that $H_h := -h^2\Delta + V$, $h = (0,1]$, has only isolated eigenvalues, each with finite degeneracy, that may accumulate only at 0 and denote them by $-\infty < E_1(h) \leqslant E_2(h) \leqslant E_3(h) \leqslant \cdots$, counting multiplicities.

(i) For any $E < 0$ define $\Omega_E := \{(q,k) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, (2\pi h)^2 |k|^2 + V(q) \leqslant E\}$ and denote by χ_E its characteristic function. Show that

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathrm{d}q\, \mathrm{d}k\; \chi_E(q,k)\big((2\pi h)^2 |k|^2 + V(q) - E\big) \;=\; -\frac{1}{15\pi^2 h^3} \int_{\mathbb{R}^3} \mathrm{d}x\, (V(x) - E)_-^{5/2}\,.$$

(ii) Let $E < 0$. Prove the bound

$$\sum_{n \text{ s.t. } E_n(h) < E} E_n(h) \;\leqslant\; -\frac{1}{15\pi^2 h^3} \int_{\mathbb{R}^3} \mathrm{d}x\, (V(x) - E)_-^{5/2} + o(h^{-3}) \qquad \text{as } \; h \to 0.$$

(*Hint:* the ingredients are: $\mathfrak{Z}(\chi_E)$, as defined in Exercise 43, as a trial density matrix for the estimate proved in Exercise 42; an appropriate $\theta > 0$ in the definition of $\psi_{q,k}$; Exercise 42; part (i) above.)