EXERCISE SHEET 6, issued on Tuesday 30 November 2010
Due: Tuesday 7 December 2010 by 8,15 a.m. in the designated "MQM" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS10_MQM.html`

Each exercise is worth a full mark as specified below. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English. The exercise marked with ★ is for extra credit.

Exercise 21 [14 points]. Consider the Hamiltonian $H = -\Delta + V$ in d dimensions and its ground state energy

$$E_0 \;=\; \inf_{\substack{\|\psi\|_2=1\\ \psi\in\mathcal{M}}} \left[\int_{\mathbb{R}^d} |\nabla\psi(x)|^2\,\mathrm{d}x + \int_{\mathbb{R}^d} V(x)|\psi(x)|^2\,\mathrm{d}x\right]$$

with $\mathcal{M} := H^1(\mathbb{R}^d)\cap\{\psi|\int V_-|\psi|^2\mathrm{d}x<\infty\}$. The potential V is assumed not to vanish almost everywhere.

(i) Let $d\geqslant 3$. Assume that $V\in L^{d/2}(\mathbb{R}^d)+L^\infty(\mathbb{R}^d)$ and $|\{x\in\mathbb{R}^d \text{ s.t. } |V(x)|\geqslant\varepsilon\}|<\infty$ $\forall\varepsilon>0$ (no assumption on the sign of V). Prove that $E_0\leqslant 0$.

(ii) Assume that $V\in L^{1+\varepsilon}(\mathbb{R}^2)+L^\infty(\mathbb{R}^2)$ for some $\varepsilon>0$ and that $V(x)\leqslant 0$. Prove that $E_0<0$. (*Hint:* a logarithmic cut-off as a trial function.)

Exercise 22 [13 points]. Consider the distribution δ in $\mathcal{D}'(\mathbb{R})$. The distribution $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\delta$ is often denoted $\delta^{(k)}$. For $k=1,2,3$ the notation δ', δ'', δ''' (respectively) is also used. $\delta^{(0)}=\delta$.

(i) Let k be a non-negative integer and $f\in C^\infty(\mathbb{R})$. Show that there exist constants c_{kj}, to determine, such that the following identity holds in $\mathcal{D}'(\mathbb{R})$:

$$f\delta^{(k)} \;=\; \sum_{j=0}^{k} c_{kj}\delta^{(j)}\,.$$

(ii) Let k be a positive integer. Show that the general solution to the distributional equation

$$x^kT \;=\; 0 \qquad\qquad (T\in\mathcal{D}'(\mathbb{R}))$$

is $T=c_0\delta+c_1\delta'+\cdots+c_{k-1}\delta^{(k-1)}$, $c_0,\ldots,c_{k-1}\in\mathbb{C}$.

(iii) Determine all solutions $T \in \mathcal{D}'(\mathbb{R})$ to the distributional differential equation

$$T' \;=\; \delta_1 - \delta_{-1} + \operatorname{sgn}(x)\mathbb{1}_{\{|x|\geqslant 1\}}$$

where δ_{x_0} is the delta distribution at x_0 and $\operatorname{sgn}(x)$ is the signum of x.

Exercise 23 [13 points]. Let d be a positive integer and let $m > 0$. Define G and G_m in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ by

$$G(x) \;:=\; \begin{cases} -\dfrac{\ln|x|}{|\mathbb{S}^1|} & d = 2 \\ \dfrac{1}{(d-2)|\mathbb{S}^{d-1}|}\,\dfrac{1}{|x|^{d-2}} & d \neq 2 \end{cases}$$

and

$$G_m(x) \;:=\; \left(\frac{1}{m^2 + (2\pi\,\cdot)^2}\right)^{\vee}(x)$$

where $|\mathbb{S}^{d-1}| = 2\pi^{d/2}/\Gamma(d/2)$ is the area of the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ and $f^{\vee}$ denotes the inverse Fourier transform of f.

(i) Prove that $\dfrac{G_m(x)}{G(x)} \xrightarrow{|x|\to 0} 1$ for $d = 3$. (*Optional*, i.e., not needed for the mark: prove that the same holds $\forall d \geqslant 2$.)

(ii) Prove that $-\dfrac{\ln G_m(x)}{m|x|} \xrightarrow{|x|\to\infty} 1$ for $d = 3$. (*Optional:* prove that the same holds in any dimension.)

(iii) Prove that $(-\Delta + m^2)G_m = \delta$ as an identity of distributions in $\mathcal{D}'(\mathbb{R}^d)$.

★ **Exercise 24 [15 points].** Consider the Hamiltonian $H = -\Delta - V$ in three dimensions where the potential V does not vanish almost everywhere, $V \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $V \geqslant 0$. Assume that some $f \in C^2(\mathbb{R}^3)$, $f \geqslant 0$, satisfies the P.D.E. $(-\Delta - V)f = 0$. Show that either $f(x) > 0$ $\forall x \in \mathbb{R}^3$ or $f(x) = 0$ $\forall x \in \mathbb{R}^3$. (*Hint:* prove that $f(x) \geqslant \frac{1}{4\pi r^2}\int_{|x-y|=r} f(y)\mathrm{d}y$ $\forall x \in \mathbb{R}^3$, e.g. by computing $\frac{\mathrm{d}}{\mathrm{d}t}\int_{|x-y|=r} f(x + t(y-x))\mathrm{d}y$, and use this inequality.)