Mathematical Quantum Mechanics

TMP Programme, Munich – Winter Term 2010/2011

EXERCISE SHEET 2, issued on Tuesday 2 November 2010
Due: Tuesday 9 November 2010 by 8,15 a.m. in the designated "MQM" box on the 1st floor
Info: `www.math.lmu.de/~michel/WS10_MQM.html`

The full mark in each exercise is 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 5. Prove that

$$\begin{aligned} \pi \int_{\mathbb{R}^3} \frac{1}{|x|}\, \psi(x)\, \mathrm{d}x &= \int_{\mathbb{R}^3} \frac{1}{|k|^2}\, \widehat{\psi}(k)\, \mathrm{d}k \\ \int_{\mathbb{R}^3} \frac{1}{|x|^2}\, \psi(x)\, \mathrm{d}x &= \pi \int_{\mathbb{R}^3} \frac{1}{|k|}\, \widehat{\psi}(k)\, \mathrm{d}k \end{aligned}$$

for all $\psi \in \mathcal{S}(\mathbb{R}^3)$.

Exercise 6. Let $\psi : \mathbb{R}^d \to \mathbb{C}$ be a measurable function such that

$$|\psi(x)| \leqslant \frac{C}{(1+|x|)^\alpha} \qquad \forall x \in \mathbb{R}^d$$

for some constants $C > 0$ and $\alpha > d$. Prove that $\widehat{\psi} \in C^k(\mathbb{R}^d)$ for every integer $k < \alpha - d$.

Exercise 7.

(i) Let $\varphi \in \mathcal{S}(\mathbb{R}^3)$. Prove that for every $a > 0$ there exists $b_a > 0$, independent of φ, such that

$$\|\varphi\|_{L^\infty(\mathbb{R}^3)} \leqslant a\|\Delta\varphi\|_{L^2(\mathbb{R}^3)} + b_a\|\varphi\|_{L^2(\mathbb{R}^3)}\,.$$

(ii) Prove that the inequality

$$\|\varphi\|_{L^\infty(\Omega)} \leqslant \|\Delta\varphi\|_{L^2(\Omega)} + 2010\,\|\varphi\|_{L^2(\Omega)}$$

cannot hold for all $\varphi \in C^\infty(\Omega)$ where $\Omega \subset \mathbb{R}^3$ is open and with compact closure.

Exercise 8.

(i) Consider the quantum coherent state $\psi_{\xi,k,\theta} \in L^2(\mathbb{R}^d)$ relative to the classical state $(\xi, k) \in \mathbb{R}^{2d}$ with variance $\theta > 0$ and consider its evolution $e^{it\Delta}\psi_{\xi,k,\theta}$ under the free propagator (see Exercise 4). Prove that $e^{it\Delta}\psi_{\xi,k,\theta} \in L^q(\mathbb{R}^d)$ $\forall t > 0$ and $\forall q \in [2, \infty]$ and

$$\|e^{it\Delta}\psi_{\xi,k,\theta}\|_q \leqslant \frac{1}{t^{d(\frac{1}{2}-\frac{1}{q})}} \|\psi_{\xi,k,\theta}\|_p \qquad (t > 0\,,\ q \in [2,\infty]\,,\ p^{-1} + q^{-1} = 1)\,.$$

In the following let $\psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$.

(ii) Consider the evolution $e^{it\Delta}\psi$ under the free propagator (see Exercise 3). Prove that

$$\|e^{it\Delta}\psi\|_2 = \|\psi\|_2 \qquad \text{and} \qquad \|e^{it\Delta}\psi\|_\infty \leqslant \frac{1}{(4\pi t)^{d/2}} \|\psi\|_1 \qquad \forall t > 0\,.$$

(iii) Deduce from (ii) that $e^{it\Delta}\psi \in L^q(\mathbb{R}^d)$ $\forall t > 0$ and $\forall q \in [2, \infty]$, with

$$\|e^{it\Delta}\psi\|_q \leqslant \frac{1}{t^{d(\frac{1}{2}-\frac{1}{q})}} \|\psi\|_1^{1-2/q}\|\psi\|_2^{2/q} \qquad (t > 0\,,\ q \in [2,\infty])\,.$$

(*Hint:* use Hölder's inequality to prove that if $\psi \in L^{p_0}(\mathbb{R}^d) \cap L^{p_1}(\mathbb{R}^d)$ for some p_0, p_1 with $1 \leqslant p_0 < p_1 \leqslant \infty$ then $\psi \in L^p(\mathbb{R}^d)$ for all $p \in [p_0, p_1]$ and

$$\|\psi\|_p \leqslant \|\psi\|_{p_0}^{\frac{\frac{1}{p}-\frac{1}{p_1}}{\frac{1}{p_0}-\frac{1}{p_1}}} \|\psi\|_{p_1}^{\frac{\frac{1}{p_0}-\frac{1}{p}}{\frac{1}{p_0}-\frac{1}{p_1}}}\,).$$

(iv) Use the Riesz-Thorin interpolation theorem (it will be discussed in class over the next lectures, you may find it stated here below) and the estimates found in (ii) to prove that for every $t > 0$ $e^{it\Delta}$ extends uniquely to a bounded linear operator $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$, $p \in [1, 2]$, $\frac{1}{p} + \frac{1}{q} = 1$, with

$$\|e^{it\Delta}\psi\|_q \leqslant \frac{1}{t^{d(\frac{1}{2}-\frac{1}{q})}} \|\psi\|_p\,.$$

THE RIESZ-THORIN THEOREM. *Let $p_0, p_1, q_0, q_1 \in [1, \infty]$ and let $\theta \in (0, 1)$. Define $p, q \in [1, \infty]$ by*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\,.$$

If T is a linear map with

$$\begin{aligned} T : L^{p_0} \to L^{q_0}\,, &\qquad \|T\|_{L^{p_0}\to L^{q_0}} = N_0\,, \\ T : L^{p_1} \to L^{q_1}\,, &\qquad \|T\|_{L^{p_1}\to L^{q_1}} = N_1\,, \end{aligned}$$

then we have

$$\|Tf\|_q \leqslant N_0^{1-\theta}\, N_1^{\theta}\, \|f\|_p$$

for all $f \in L^{p_0} \cap L^{p_1}$. Hence T extends uniquely to a continuous map from L^p to L^q with $\|T\|_{L^p\to L^q} \leqslant N_0^{1-\theta}\, N_1^{\theta}$.