EXERCISE SHEET 1, issued on Tuesday 26 October 2010
Due: Tuesday 2 November 2010 by 8,15 a.m. in the designated “MQM” box on the 1st floor
Info: www.math.lmu.de/~michel/WS10_MQM.html

The full mark in each exercise is 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in the solutions either in German or in English.

Exercise 1. Let $\mathcal{H}$ be a complex Hilbert space with scalar product $\langle\cdot,\cdot\rangle$.

(i) Let $\widetilde{\mathcal{H}}$ be a dense subspace of $\mathcal{H}$ and let P and Q be two operators on $\mathcal{H}$ mapping $\widetilde{\mathcal{H}}$ into itself and such that $\langle\phi,P\xi\rangle=\langle P\phi,\xi\rangle$ and $\langle\phi,Q\xi\rangle=\langle Q\phi,\xi\rangle$ $\forall\phi,\xi\in\widetilde{\mathcal{H}}$. Fix $\psi\in\widetilde{\mathcal{H}}$. Define

$$\langle P\rangle\;:=\;\langle\psi,P\psi\rangle\,,\qquad\qquad \Delta P\;:=\;\sqrt{\langle(P-\langle P\rangle)^2\rangle}$$

and similarly for Q. Prove that

$$(\Delta Q)(\Delta P)\;\geqslant\;\frac{1}{2}\big|\langle[Q,P]\rangle\big| \tag{$*$}$$

where $[Q,P]:=QP-PQ$ is the commutator.

(ii) Prove that equality in $(*)$ holds if and only if $(Q-\langle Q\rangle)\psi=i\lambda(P-\langle P\rangle)\psi$ for some $\lambda\in\mathbb{R}$.

(iii) Assume that $[Q,P]=i\mathbb{1}$ (as an identity on $\widetilde{\mathcal{H}}$). Prove that it cannot be that both Q and P are bounded operators on $\widetilde{\mathcal{H}}$. (*Hint:* compute $[Q^n,P]$ for positive integer n's.)

Now choose $\mathcal{H}=L^2(\mathbb{R})$, $\widetilde{\mathcal{H}}=\mathcal{S}(\mathbb{R})$, the class of Schwarz functions[1] on $\mathbb{R}$, Q as the operator of multiplication by x, and $P=-i\frac{\mathrm{d}}{\mathrm{d}x}$.

(iv) Deduce from $(*)$ that

$$(\Delta Q)(\Delta P)\;\geqslant\;\frac{1}{2}\qquad\qquad(\textit{Heisenberg's uncertainty relation}).$$

(v) Prove that equality in Heisenberg's uncertainty relation holds if and only if ψ is a function that, apart from a translation and a phase, has the form $\psi(x)=c\,e^{-ax^2}$ for some $c\in\mathbb{R}$ and $a>0$.

(vi) Prove that there is *no* upper bound to the quantity $\big\langle\psi,(Q-\langle\psi,Q\psi\rangle)^2\psi\big\rangle\big\langle\psi,(P-\langle\psi,P\psi\rangle)^2\psi\big\rangle$ uniformly in ψ.

[1] Recall that by definition $\mathcal{S}(\mathbb{R})$ is the space of infinitely differentiable functions $f:\mathbb{R}\to\mathbb{C}$ such that

$$\sup_{x\in\mathbb{R}}\Big|x^n\frac{\mathrm{d}^m}{\mathrm{d}x^m}f(x)\Big|\;<\;\infty\qquad\qquad\text{for all non-negative integers } m \text{ and } n\,.$$

Exercise 2. Let A be a symmetric and positive definite $d \times d$ real matrix and $b \in \mathbb{R}^d$. Define

$$f(x) \;:=\; e^{-x\cdot Ax + b\cdot x}\,, \qquad x \in \mathbb{R}^d\,.$$

Show that the Fourier transform $\widehat{f}$ of f is given by

$$\widehat{f}(k) \;=\; \frac{1}{2^{d/2}\sqrt{\det A}}\, e^{-\frac{1}{4}(k+ib)\cdot(A^{-1}(k+ib))} \qquad (k \in \mathbb{R}^d)$$

with the convention $\widehat{f}(k) = (2\pi)^{-d/2}\int e^{-ikx} f(x)\mathrm{d}x$, or

$$\widehat{f}(k) \;=\; \frac{\pi^{d/2}}{\sqrt{\det A}}\, e^{-\frac{1}{4}(2\pi k+ib)\cdot(A^{-1}(2\pi k+ib))} \qquad (k \in \mathbb{R}^d)$$

with the convention $\widehat{f}(k) = \int e^{-2\pi ikx} f(x)\mathrm{d}x$.

Exercise 3. For every $t \in \mathbb{R}$ define the bounded linear operator $e^{it\Delta} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$e^{it\Delta} \;:=\; \mathcal{F}^{-1} \circ \mathcal{M}_{\exp(-it|2\pi k|^2)} \circ \mathcal{F}\,,$$

where $\mathcal{F}$ is the Fourier transform operator on L^2 and $\mathcal{M}_{\exp(-it|2\pi k|^2)} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the multiplication operator by the function $k \mapsto e^{-it|2\pi k|^2}$.

(i) Prove that $\forall \psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ one has

$$(e^{it\Delta}\psi)(x) \;=\; e^{-i\frac{\pi}{4}d} \int_{\mathbb{R}^d} \frac{e^{i\frac{|x-y|^2}{4t}}}{(4\pi t)^{d/2}}\,\psi(y)\,\mathrm{d}y$$

almost everywhere on $\mathbb{R}^d$.

(ii) Using (i) prove that

$$\left\| e^{it\Delta}\psi \;-\; e^{-i\frac{\pi}{4}d}\int_{\mathbb{R}^d} \frac{e^{i\frac{|\,\cdot\,-y|^2}{4t}}\, e^{-\frac{|y|^2}{R^2}}}{(4\pi t)^{d/2}}\,\psi(y)\,\mathrm{d}y \right\|_{L^2} \xrightarrow{R\to\infty} 0 \qquad \forall \psi \in L^2(\mathbb{R}^d)\,.$$

(iii) Prove that, $\forall \psi \in \mathcal{S}(\mathbb{R}^d)$, $\phi(t,x) := (e^{it\Delta}\psi)(x)$ is a solution to the partial differential equation

$$i\partial_t \phi(t,x) \;=\; -\Delta_x \phi(t,x)$$

in $C^\infty((\mathbb{R}\backslash\{0\})_t \times \mathbb{R}^d_x)$.

Exercise 4.

(i) For $q,p \in \mathbb{R}^d$ and $\theta > 0$ prove that the function $\psi_{q,p,\theta} : \mathbb{R}^d \to \mathbb{C}$ defined by

$$\psi_{q,p,\theta}(x) \;:=\; \frac{1}{(\theta\sqrt{\pi})^{d/2}}\, e^{ip\cdot x}\, e^{-\frac{|x-q|^2}{2\theta^2}}$$

satisfies

$$\|\psi_{q,p,\theta}\|_2 \;=\; 1\,, \qquad \langle \psi_{q,p,\theta}, x\,\psi_{q,p,\theta}\rangle \;=\; q\,, \qquad \langle \psi_{q,p,\theta}, -i\nabla_x\,\psi_{q,p,\theta}\rangle \;=\; p$$

and makes Heisenberg's uncertainty relation of Exercise 1 part (iv) an equality. (Such a $\psi_{q,p,\theta}$ is called the *coherent state* for the classical state $(q,p) \in \mathbb{R}^{2d}$ with *variance* θ.)

(ii) Using the definition given in Exercise 3, compute $e^{it\Delta}\psi_{q,p,\theta}$ explicitly.

(iii) Show that, apart from an irrelevant phase factor, $e^{it\Delta}\psi_{q,p,\theta}$ is also a coherent state of the form $\psi_{q(t),p(t),\theta(t)}$ where $q(t) = q + 2pt$, $p(t) = 2p$, $\theta(t) = \sqrt{\theta^2 + 4t^2/\theta^2}$.