Spring term 2014 / *Sommersemester 2014*

Mathematical Statistical Physics – Final exam, 11.7.2014

Mathematische Statistische Physik – Endklausur, 11.7.2014

Name:/*Name*: ______________________ **Pseudonym:**/*Pseudonym:* __________

Matriculation number:/*Matrikelnr.:* ______________ **Semester:**/*Fachsemester:* __________

Degree course:/*Studiengang:* ❑ Diplom ❑ Bachelor, PO ______ ❑ Lehramt Gymnasium (modularisiert)
❑ TMP ❑ Master, PO ______ ❑ Lehramt Gymnasium (nicht modul.)
❑ ______________________

Major:/*Hauptfach:* ❑ Mathematik ❑ Wirtschaftsm. ❑ Informatik ❑ Physik ❑ Statistik ❑ ______

Minor:/*Nebenfach:* ❑ Mathematik ❑ Wirtschaftsm. ❑ Informatik ❑ Physik ❑ Statistik ❑ ______

Credits needed for:/*Anrechnung der Credit Points für das:* ❑ Hauptfach ❑ Nebenfach

Extra solution sheets submitted:/*Zusätzlich abgegebene Lösungsblätter:* ❑ Yes ❑ No

problem	**1**	**2**	**3**	**4**	**5**	**6**	$\sum$
total marks	**10**	**10**	**10**	**10**	**10**	**10**	60
scored marks							

FINAL MARK	

INSTRUCTIONS:

- This booklet is made of sixteen pages, including the cover, numbered from 1 to 16. The test consists of six problems. Each problem is worth 10 marks. You are free to work on any problem and to collect partial credits.
- The only material that you are allowed to use is black or blue pens/pencils and one hand-written, two-sided, A4-size "cheat sheet" (Spickzettel).
- Raise up your hand to request extra sheets or scratch paper. You are not allowed to use your own paper.
- Results discussed in the lectures can be cited and used without justification. Unless explicitly stated, you cannot cite without proof results from the homework.
- Work individually. Write with legible handwriting. You may hand in your solution in German or in English. Put your name on every sheet you hand in.
- You have 165 minutes.

GOOD LUCK!

Fill in the form here below only if you need the certificate (Schein).

Dieser Leistungsnachweis entspricht auch den Anforderungen
nach § Abs. Nr. Buchstabe LPO I
nach § Abs. Nr. Buchstabe LPO I

UNIVERSITÄT MÜNCHEN

ZEUGNIS

Der / Die Studierende der ______

Herr / Frau ______ *aus* ______

geboren am ______ *in* ______ *hat im* **SoSe**______ *–Halbjahr* **2014**

meine Übungen **zur Mathematischen Statistischen Physik** ______

mit ______ *besucht.*

Er / Sie hat ______

schriftliche Arbeiten geliefert, die mit ihm / ihr besprochen wurden. ______

MÜNCHEN, *den* **11 Juli 2014**

Name

PROBLEM 1. (10 marks)

Consider

- the C^*-algebra $\mathcal{A} = \mathcal{M}(n \times n, \mathbb{C})$, $n \in \mathbb{N}$,
- a Hamiltonian $H = H^* \in \mathcal{A}$,
- the one-parameter group $\{\tau^t \,|\, t \in \mathbb{R}\}$ of $*$-automorphisms $A \mapsto \tau^t(A) := e^{\mathrm{i}tH} A e^{-\mathrm{i}tH}$ of $\mathcal{A}$,
- a state ω on $\mathcal{A}$,
- $\beta \in \mathbb{R}$.

Denote by $\omega_{\beta H}$ the Gibbs state at inverse temperature β. Prove the following:

1. If $\omega = \omega_{\beta H}$, then ω is a (τ^t, β)-KMS state.
2. If ω is a (τ^t, β)-KMS state, then $\omega = \omega_{\beta H}$.

SOLUTION:

SOLUTION TO PROBLEM 1 (CONTINUATION):

Name

PROBLEM 2. (10 marks)

Consider the C^*-algebra $\mathcal{A} = \mathcal{M}(2 \times 2, \mathbb{C})$ and the state ω_μ represented by the density matrix

$$\rho_\mu \; = \; \begin{pmatrix} \mu & 0 \\ 0 & 1-\mu \end{pmatrix}, \qquad \mu \in \left[0, \frac{1}{2}\right].$$

Find, for each μ, an explicit GNS representation $(\mathcal{H}_\mu, \pi_\mu, \Omega_\mu)$ of $\mathcal{A}$ associated to the state ω_μ.

SOLUTION:

SOLUTION TO PROBLEM 2 (CONTINUATION):

Name

PROBLEM 3. (10 marks)

Consider a one-dimensional quantum spin chain. For each $x \in \mathbb{Z}$, let $\mathcal{H}_x \simeq \mathbb{C}^n$ for a fixed $n \geqslant 2$, and let $\mathcal{A}_{\mathbb{Z}}$ be the usual quasi-local algebra built upon $\mathcal{A}_{\{x\}} = \mathcal{B}(\mathcal{H}_x)$. Let $(\Lambda_m)_{m\in\mathbb{N}}$ be the sequence $\Lambda_m = [-m, m] \cap \mathbb{Z}$. Consider:

- unitary elements $U_x \in \mathcal{A}_{\{x\}}$ and the associated map

$$\alpha_\Lambda(A) := \left(\otimes_{x\in\Lambda} U_x^*\right) A \left(\otimes_{y\in\Lambda} U_y\right), \qquad A \in \mathcal{A}_\Lambda;$$

- the local Hamiltonian H_Λ, given by a two-body interaction

$$H_\Lambda = \sum_{x,y\in\Lambda} J(x,y)\Phi_{x,y}\,,$$

  where $\Phi_{x,y} \in \mathcal{A}_{\{x\}\cup\{y\}}$; we shall assume that the associated dynamics τ^t exists on $\mathcal{A}_{\mathbb{Z}}$.

1. Prove that there exists an automorphism α of $\mathcal{A}_{\mathbb{Z}}$ such that

$$\lim_{m\to\infty} \alpha_{\Lambda_m}(A) = \alpha(A), \qquad A \in \mathcal{A}_{\text{loc}}.$$

2. Assume that $\|\Phi_{x,y}\| \leqslant 1$ and that there exists $C < \infty$ such that

$$\sup_{x\in\mathbb{Z}} \sum_{y\in\mathbb{Z}} |J(x,y)||x-y| = C.$$

   Assume moreover that $\alpha(\Phi_{x,y}) = \Phi_{x,y}$ for all $(x,y) \in \mathbb{Z}\times\mathbb{Z}$. Prove that if ω is a (τ^t, β)-KMS state for a $\beta \in (0,\infty)$, then $\omega \circ \alpha = \omega$.

SOLUTION:

SOLUTION TO PROBLEM 3 (CONTINUATION):

Name

PROBLEM 4. (10 marks)

For this problem, you may assume results proved in the homework.

Let $M := \{1, 2, \ldots, m\}$. Consider the following generalisation of the classical 1D Ising Model discussed in class. The model consists of $N+1$ sites where at each site i the configuration is given by $S_i \in M$. A configuration of the chain is given by $S \colon \{1, \ldots, N+1\} \to M$. The energy functional is given by

$$H_N(S) \;:=\; \sum_{i=1}^{N} h(S_i, S_{i+1})$$

for some function $h \colon M \times M \to \mathbb{R}$. We consider the Dirichlet boundary condition obtained by fixing the values of S_1 and S_{N+1}. For any $(s_L, s_R) \in M \times M$, denote $C(s_L, s_R) := \{S \,|\, S_1 = s_L, S_{N+1} = s_R\}$. The specific free energy is given by

$$f_N(\beta, s_L, s_R) \;=\; \frac{\log Z_N(\beta, s_L, s_R)}{N+1}, \qquad Z_N(\beta, s_L, s_R) = \sum_{S \in C(s_L, s_R)} e^{-\beta H_N(S)}\,.$$

1. Prove that the thermodynamic limit

   $$f(\beta, s_L, s_R) \;=\; \lim_{N\to\infty} f_N(\beta, s_L, s_R)$$

   exists.

2. Prove that it is independent of the boundary condition (s_L, s_R).

(*Hint:* Introduce a transfer matrix to represent $Z_N(\beta, s_L, s_R) = \langle v_L, T_\beta^N v_R\rangle$ and use the specific properties of its entries.)

SOLUTION:

SOLUTION TO PROBLEM 4 (CONTINUATION):

Name

PROBLEM 5. (10 marks)

Let $\mathcal{A}$ be a C^*-algebra with unit, $\{\tau^t \mid t \in \mathbb{R}\}$ be a one-parameter strongly continuous group of $*$-automorphisms of $\mathcal{A}$, and $\beta \in (0, \infty)$. Assume that, for each $n \in \mathbb{N}$, $\{\tau_n^t \mid t \in \mathbb{R}\}$ is a one-parameter strongly continuous group of $*$-automorphisms of $\mathcal{A}$ such that $\|\tau_n^t(A) - \tau^t(A)\| \xrightarrow{n\to\infty} 0$ $\forall A \in \mathcal{A}$ and $\forall t \in \mathbb{R}$, and that $(\omega_n)_{n=1}^\infty$ is a sequence of (τ_n^t, β)-KMS states on $\mathcal{A}$.

1. Argue that there exists a limiting state ω on $\mathcal{A}$ (a keyword is sufficient here).
2. Prove that ω is a (τ^t, β)-KMS state.

(*Hint:* You may use the following result: Let δ, respectively δ_n, be the generators of τ^t, respectively τ_n^t. For each $A \in D(\delta)$, there is a sequence $(A_n)_{n=0}^\infty$ such that $A_n \in D(\delta_n)$ and

$$A_n \longrightarrow A, \qquad \delta_n(A_n) \longrightarrow \delta(A)$$

as $n \to \infty$.)

SOLUTION:

SOLUTION TO PROBLEM 5 (CONTINUATION):

Name

PROBLEM 6. (10 marks) – SHORT QUESTIONS

Answer to each question with a YES or a NO. Marking scheme: 1 mark for each correct answer, 0 marks for each unanswered question, -1 mark for each wrong answer.

6.1 Consider the quasi-free state ω_ρ on the Weyl algebra with one-particle Hilbert space $\mathcal{H}$ associated to the operator $\rho \geqslant 0$ on $\mathcal{H}$. Let the dynamics be given by $\tau^t(W(f)) = W(e^{\mathrm{i}tH}f)$, and assume that $\rho = e^{-\mathrm{i}tH}\rho e^{\mathrm{i}tH}$. Is the dynamics unitarily implementable in the GNS representation associated to ω_ρ?

❏ YES ❏ NO

6.2 Let $H_\Lambda = \sum_{(x,y)\in\mathcal{E}_\Lambda}(\lambda(\sigma_x^1\sigma_y^1 + \sigma_x^2\sigma_y^2) + \sigma_x^3\sigma_y^3)$, $|\lambda| < 1$ for $\Lambda \subset \mathbb{Z}^2$, where $\sigma^\cdot$ are the Pauli matrices. Consider the symmetry $\sigma_x^1 \mapsto \sigma_x^1$, $\sigma_x^2 \mapsto \sigma_x^2$, $\sigma_x^3 \mapsto -\sigma_x^3$ for all $x \in \mathbb{Z}^2$. Does Mermin-Wagner's theorem show the absence of symmetry breaking in this case?

❏ YES ❏ NO

6.3 Does an ideal Bose gas with relativistic energy-momentum dispersion relation $\epsilon(p) = |p|$, i.e. $h = \sqrt{|-\mathrm{i}\nabla|}$ on $L^2(\mathbb{R}^2)$, exhibit Bose-Einstein condensation?

❏ YES ❏ NO

6.4 Consider a C^*-dynamical system $(\mathcal{A}, \tau^t)$, a (τ^t, β)-KMS state ω with $\beta \in (0, \infty)$ on $\mathcal{A}$, and a $*$-automorphism α on $\mathcal{A}$. Is it true that, if the dynamics τ^t is α-invariant, so is the state ω?

❏ YES ❏ NO

6.5 On the C^*-algebra $\mathcal{A} = \mathcal{M}(n \times n, \mathbb{C})$, consider a Hamiltonian $H = H^*$ and a state ω_ρ given by the density matrix ρ. Assume that $\omega_\rho(X^*[H, X]) \geqslant -2014$ for all $X \in \mathcal{A}$. Do ρ and H commute?

❏ YES ❏ NO

6.6 Let $\mathcal{A}$ be a C^*-algebra and π_1, π_2 be unitarily equivalent representations. Does this imply that the spectrum of $\pi_1(A)$ is equal to the spectrum of $\pi_2(A)$ for all $A \in \mathcal{A}$?

❏ YES ❏ NO

6.7 Let $\mathcal{A} = C^0([0,1])$ and $\mathcal{A} \ni f \colon x \mapsto \exp(x)$. Does 2014 belong to the spectrum of f?

❏ YES ❏ NO

6.8 Consider the family $\mathcal{N}(\mathcal{A}) := \{A \in \mathcal{A} \,|\, AA^* = A^*A\}$ of all normal elements of a C^*-algebra $\mathcal{A}$. Is $\mathcal{N}(\mathcal{A})$ necessarily a $*$-subalgebra of $\mathcal{A}$?

❏ YES ❏ NO

6.9 Consider a quantum spin system on $\Gamma = \mathbb{Z}^2$. For boxes $\Lambda_L = [-L \ldots L]^2$ let f_β^L be the Fourier transform of $\omega_\beta^L(S_0^2 S_x^2)$, where ω_β^L is the Gibbs state of the system on Λ_L. Assume that f_β^L is real-valued and

$$f_\beta^L(k) \leqslant \frac{C}{L^2\beta}|k|^{-1/2}$$

for a positive constant C. Is it true that the model exhibits long-range order in the sense that $\liminf_{L\to\infty} L^{-2} \sum_{x\in\mathbb{Z}^2} \omega_\beta^L(S_0^2 S_x^2) > 0$?

❏ YES ❏ NO

6.10 On a Hilbert space $\mathcal{H}$, assume that the time-dependent Hamiltonian $H(t) = H(t)^*$ is such that $H(0) = H(T)$. Let ω be an arbitrary state of the system. Can the energy of the system at $t = T$ be strictly smaller than at $t = 0$?

❏ YES ❏ NO