HOMEWORK ASSIGNMENT – WEEK 10
Hand-in deadline: Fri 20 June by 12 p.m. in the "MSP" drop box.
Info: `www.math.lmu.de/~michel/SS14_MSP.html`

Exercise 27. (A theorem about matrices with positive entries.)

Define the following relation for finite-dimensional matrices (and vectors as a special case):

$$(a_{ij}) \succ (\succeq) 0 \quad :\Leftrightarrow \quad a_{ij} > (\geq) 0 \quad \forall\, i,j$$

and, correspondingly,

$$(a_{ij}) \succ (\succeq)(b_{ij}) \quad :\Leftrightarrow \quad (a_{ij}) - (b_{ij}) \succ (\succeq) 0\,.$$

Let $n \in \mathbb{N}$ and let $A \in \mathcal{M}(n \times n, \mathbb{C})$ be such that $A \succ 0$.

(i) Prove that there exist $\lambda_0 > 0$ and $x_0 \in \mathbb{R}^n$, $x_0 \succ 0$, such that $Ax_0 = \lambda_0 x_0$.
(*Hint:* One route you may take is to apply a suitable fixed point argument. Alternatively, you may define the set $\Lambda := \{\mu \geq 0 \,|\, \exists\, x \succ 0 \text{ such that } Ax \succeq \mu x\}$ and find its supremum.)

(ii) Prove that if $\lambda \neq \lambda_0$ is a (possibly complex) eigenvalue of A, then $|\lambda| < \lambda_0$.

(iii) Prove that the eigenvalue λ_0 is simple (i.e., its algebraic multiplicity is 1).

Exercise 28. (Application of Ex. 27: irreducible, aperiodic Markov processes.)

Consider a n-dimensional transition matrix P for a Markov process that is *irreducible* and *aperiodic* (according to the definitions given in class).

(i) Prove that there exists $t \in \mathbb{N}$ such that $P^t \succ 0$.

(ii) Prove that the eigenvalue λ_0 of P^t determined by means of the theorem proved in Exercise 27 is actually $\lambda_0 = 1$.

(iii) Prove that for every $x \in \mathbb{R}^n$ such that $x \succ 0$ and $\sum\limits_{i=1}^{n} x_i = 1$ one has

$$\|P^n x - x_0\| \;<\; C\mu^n$$

for some constants $C > 0$ and $\mu \in (0,1)$, where $\mathbb{R}^n \ni x_0 \succ 0$, $P^t x_0 = x_0$ (as in Exercise 27), whence in particular

$$\lim_{n\to\infty} P^n x \;=\; x_0$$

in the vector-norm sense. (In fact this holds also in the matrix-norm sense, because of the finite-dimensional setting.)

Exercise 29. (Mean field and the gap equation.)

Consider:

- a quantum spin system of spin-s particles on a finite lattice $\Lambda \subset \mathbb{Z}^d$ (thus, for each $x \in \Lambda$ the algebra $\mathcal{A}_{\{x\}}$ consists of the $n \times n$ matrices, $n = 2s + 1 \geq 2$);
- a one-site self-adjoint matrix A, a two-site self-adjoint matrix B, and the Ising-type Hamiltonian
$$H_\Lambda := \sum_{x \in \Lambda} A_x + \sum_{\substack{x,y \in \Lambda, \\ |x-y|=1}} B_{xy}$$
where $A_x \in \mathcal{A}_{\{x\}}$ and $B_{xy} \in \mathcal{A}_{\{x,y\}}$ are copies, respectively, of A and B;
- a state $\omega_{(\rho)}$ on $\mathcal{A}_\Lambda$ (customarily referred to as "product" or "mean field" state) whose density matrix is $\rho^{\otimes |\Lambda|}$, where ρ is a given one-site density matrix ρ;
- the free energy functional F_β ($\beta > 0$) given, on every state ω on $\mathcal{A}_\Lambda$, by
$$F_\beta(\omega) := \frac{1}{\beta} S(\omega) - \omega(H_\Lambda)\,,$$
where $S(\omega)$ is the entropy of ω (see Exercise 17).

(i) Prove that the limit
$$f_\beta(\rho) := \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} F_\beta(\omega_{(\rho)})$$
exists and can be written as
$$f_\beta(\rho) = -\frac{1}{\beta} \mathrm{Tr}(\rho \ln \rho) - \mathrm{Tr}(\rho H_\rho)\,,$$
where
$$H_\rho = A + d \cdot \mathrm{Tr}_2\left((\mathbb{1} \otimes \rho) B\right)\,.$$
Here Tr_2 denotes the partial trace w.r.t. the second factor of $\mathcal{H}_{xy} \cong \mathbb{C}^n \otimes \mathbb{C}^n$. The limit $|\Lambda| \to \infty$ is meant to be, as usual in this context, a limit over an arbitrary sequence $(\Lambda_N)_{N=1}^\infty$ of finite lattices such that

- $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \cdots$, $|\Lambda_N| \xrightarrow{N \to \infty} \infty$,
- $|\Lambda_N^0| / |\Lambda_N| \xrightarrow{N \to \infty} 1$, where $|\Lambda_N^0|$ = number of sites of Λ_N that are at a distance > 1 away from the boundary of Λ_N.

(ii) Use the energy/entropy balance inequality to prove that any ρ that maximises $f_\beta(\rho)$ is a solution to
$$\rho = \frac{e^{-\beta H_\rho}}{\mathrm{Tr}(e^{-\beta H_\rho})} \qquad \text{(the "gap equation")}\,.$$