HOMEWORK ASSIGNMENT – WEEK 08-09
Hand-in deadline: Thu 12 June by 12 p.m. in the "MSP" drop box.
Info: `www.math.lmu.de/~michel/SS14_MSP.html`

Exercise 24. (Thermodynamic limit of a free Bose gas)

Consider

- a bounded open domain Λ in $\mathbb{R}^d$, $d \in \mathbb{N}$,
- the domains $\Lambda_L := \{x \in \mathbb{R}^d \,|\, x/L \in \Lambda\}$, for every $L > 0$,
- the free particle Hamilton operators $h_L = -\Delta$ on $L^2(\Lambda_L)$ with Dirichlet boundary conditions

and use the fact that the spectrum of h_1 consists of a sequence of eigenvalues $0 \leq \varepsilon_1 < \varepsilon_2 \leq \varepsilon_3 \cdots$ accumulating at infinity according to the Weyl law:

$$\lim_{\varepsilon \to \infty} \frac{N(\varepsilon)}{\varepsilon^{d/2}} = \text{const},$$

where $N(\varepsilon)$ is the number of eigenvalues smaller or equal to ε (for simplicity we assume $\varepsilon_1 < \varepsilon_2$). In this problem and in the following you are asked to study, for $\beta > 0$, $0 < z \leqslant 1$, the quantity

$$\rho_L(z, \beta) \;:=\; \frac{1}{|\Lambda_L|} \operatorname{Tr} \frac{z\, e^{-\beta h_L}}{1 - z e^{-\beta h_L}}$$

when $L \to \infty$. To this aim, fix $\overline{\rho} > 0$ and define z_L to be the unique solution to

$$\overline{\rho} \;=\; \rho_L(z_L, \beta)\,, \qquad z_L := e^{\beta \mu_L}\,, \qquad \mu_L < h_L\,.$$

In terms of the eigenfunctions $\psi_N^{(L)}$ of h_L, namely, $h_L \psi_N^{(L)} = \varepsilon_N^{(L)} \psi_N^{(L)}$, $N = 1, 2, 3, \ldots$, write

$$\rho_L(z_L, \beta) \;=\; \sum_{N=1}^{\infty} \rho_L^{(N)}(z_L, \beta)\,, \qquad \rho_L^{(N)}(z_L, \beta) := \frac{1}{|\Lambda_L|} \left\langle \psi_N^{(L)}, \frac{z_L\, e^{-\beta h_L}}{1 - z_L e^{-\beta h_L}} \psi_N^{(L)} \right\rangle$$

and prove that

$$\lim_{L \to \infty} \rho_L^{(N)}(z_L, \beta) \;=\; 0 \qquad \text{when } N > 1\,.$$

Exercise 25. (Follow-up to Exercise 24)

With respect to the assumptions and notation of Exercise 24, prove that

$$\lim_{L \to \infty} \sum_{n=2}^{\infty} \rho_L^{(n)}(z_L, \beta) \;=\; C \;<\; \infty$$

where the constant C is *independent* of $\overline{\rho}$ (recall that in class it was claimed that the C amounts exactly to $\rho_c(\beta)$), and thus that

$$\lim_{L \to \infty} \rho_L^{(1)}(z_L, \beta) \;>\; 0\,.$$

Exercise 26. (Bose-Einstein statistics for Weyl's operators)

Recall that in class the KMS states for the free Bose gas were determined in terms of creation and annihilation operators. This required a *regular* representation. In this exercise you are asked to re-do the argument directly for the Weyl's operators, under the assumption that the state is *quasi-free*. To this aim, consider

- the one-particle Hilbert space $\mathfrak{h}$,
- the one-particle Hamiltonian h on $\mathfrak{h}$ such that $h > c\mathbb{1}$ for some $c > 0$,
- the Weyl operator $W(f)$ as in class, and in particular $\tau^t(W(f)) = W(e^{\mathrm{i}\,th}f)$, $t \in \mathbb{R}$,
- a (τ^t, β)-KMS state ω ($\beta > 0$) which is *quasi-free* and, correspondingly,

$$\xi(t) \;:=\; \omega(W(-f)\tau^t W(f))\,.$$

Rewrite ξ as $\xi(t) = \omega(W(\eta))F(\eta, t)$ for some $\eta \in \mathfrak{h}$ that depends on t and f and for some $F(\eta, t)$ that is analytic in t. Use the analytic continuation of F to complex-valued arguments to define

$$\xi(\mathrm{i}\beta) \;:=\; \omega(W(\eta))F(\eta, \mathrm{i}\beta)$$

and use the KMS condition to determine $\omega(W(\eta))$.