HOMEWORK ASSIGNMENT – WEEK 06

Hand-in deadline: Thu 22 May by 12 p.m. in the "MSP" drop box.

Rules: Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.

Info: `www.math.lmu.de/~michel/SS14_MSP.html`

Exercise 17.

Consider the Hilbert space $\mathcal{H} = \mathbb{C}^n$ ($n \in \mathbb{N}$) and the C^*-algebra $\mathcal{A} = \mathcal{M}(n, \mathbb{C})$.

(i) Let $A, B \in \mathcal{A}$ be such that $A > \mathbb{O}$ and $B > \mathbb{O}$, and let $f \in C^1((0, +\infty), \mathbb{R})$ be convex. Prove that

$$\mathrm{Tr}\big(f(A) - f(B) - (A - B)f'(B)\big) \;\geqslant\; 0$$

and prove that if furthermore f is strictly convex then the "=" sign in the above inequality holds if and only if $A = B$.

(ii) Given $\beta > 0$, $H = H^* \in \mathcal{A}$, ad a state ω on $\mathcal{A}$, define

$$F_\beta(\omega) \;:=\; \frac{1}{\beta} S(\omega) - \omega(H)$$

where $S(\omega)$ is the entropy of ω. Prove that $F_\beta(\omega)$ has a unique maximiser over the set of states on $\mathcal{A}$ given precisely by the Gibbs state at inverse temperature β, i.e., the state $\omega_{\rho_{\beta H}}$ defined by

$$\rho_{\beta H} \;:=\; \frac{\mathrm{e}^{-\beta H}}{\mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right)}\,.$$

Compute $F_\beta(\omega_{\rho_{\beta H}})$.

(*Hint:* Prove that $F_\beta(\omega_\rho) = \frac{1}{\beta} \log \mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right) - \frac{1}{\beta} \mathrm{Tr}\left(\rho \log \rho - \rho \log \rho_{\beta H}\right)$ and use (i) with the popular choice $f(t) = t \ln t$.)

(iii) Given $H = H^* \in \mathcal{A}$ consider the one-parameter group $\{\alpha_t \,|\, t \in \mathbb{R}\}$ of $*$-automorphisms $A \mapsto \alpha_t(A) := e^{\mathrm{i}tH} A e^{-\mathrm{i}tH}$ of $\mathcal{A}$, and let ω be a state on $\mathcal{A}$ and $\beta \in \mathbb{R}$. Prove that ω is a (α_t, β)-KMS state *if and only if* $\omega = \omega_{\rho_{\beta H}}$, the Gibbs state at inverse temperature β.

Exercise 18. Consider

- the quasi-local UHF algebra $(\mathcal{A}, (\mathcal{A}_\Lambda)_{\Lambda \in \mathcal{F}(\mathbb{Z}^d)})$ associated with an infinite quantum spin system on $\mathbb{Z}^d$ ($d \in \mathbb{N}$, $\mathcal{F}(\mathbb{Z}^d)$ is the collection of the finite subsets of $\mathbb{Z}^d$),

- an interaction $\Phi : \mathcal{F}(\mathbb{Z}^d) \to \mathcal{A}$ that is bounded, in the sense that

$$\|\Phi\| \;:=\; \sup_{x\in\mathbb{Z}^d} \sum_{\substack{\Lambda \\ \Lambda\in\mathcal{F}(\mathbb{Z}^d) \\ \Lambda\ni x}} \|\Phi(\Lambda)\| \;<\; \infty\,,$$

  and has a finite range, in the sense that $\exists\, R_\Phi \geqslant 1$ such that $\Phi(\Lambda) = \mathbb{O}$ if $\mathrm{diam}(\Lambda) > R_\Phi$, where $\mathrm{diam}(\Lambda) := \sup_{x,y\in\Lambda} |x-y|$,

- the collection $(H_\Lambda)_{\Lambda\in\mathcal{F}(\mathbb{Z}^d)}$ of local Hamiltonians $H_\Lambda := \sum_{X\subset\Lambda} \Phi(X)$.

(i) Let $A \in \mathcal{A}_{\mathrm{loc}} := \bigcup_{\Lambda\in\mathcal{F}(\mathbb{Z}^d)} \mathcal{A}_\Lambda$. Prove that the limit

$$\delta(A) \;:=\; \lim_{\substack{\Lambda\to\infty \\ \Lambda\in\mathcal{F}(\mathbb{Z}^d)}} \mathrm{i}\,[H_\Lambda, A]$$

exists in $\mathcal{A}_{\mathrm{loc}}$ and defines a symmetric derivation with domain $\mathcal{D}(\delta) = \mathcal{A}_{\mathrm{loc}}$ such that $\delta(\mathcal{D}(\delta)) \subset \mathcal{D}(\delta)$. (The limit $\Lambda\to\infty$ is meant as follows: for every sequence $\Lambda_1 \subset \Lambda_2 \subset \cdots$ in $\mathcal{F}(\mathbb{Z}^d)$ such that $\bigcup_{n=1}^\infty \Lambda_n = \mathbb{Z}^d$, $\lim_{n\to\infty} \mathrm{i}\,[H_{\Lambda_n}, A]$ converges and is independent of the choice of the sequence.)

(ii) Prove that for every $A \in \mathcal{A}_{\mathrm{loc}}$ and for sufficiently small $t\in\mathbb{R}$ the series

$$\alpha_t(A) \;\equiv\; e^{t\delta}(A) \;:=\; \sum_{n=0}^{\infty} \frac{t^n}{n!}\,\delta^n(A)$$

is norm-convergent in $\mathcal{A}$ and therefore the function $t \mapsto e^{t\delta}(A)$ is analytic.
(*Hint:* argue that $\delta^n(A) \in \mathcal{A}_{\mathrm{loc}}$ and find a control $\|\delta^n(A)\| \leqslant c(n, \|\Phi\|)\,\|A\|$ by estimating conveniently all commutators; the inequality $a^n \leqslant n!\,b^{-n} e^{ab}$ $(a,b>0,\ n\in\mathbb{N})$ may be useful.)

Exercise 19. (Follow-up to Exercise 18)

(i) Prove that the map $\alpha_t : \mathcal{A}_{\mathrm{loc}} \to \mathcal{A}$ satisfies, $\forall A, B \in \mathcal{A}_{\mathrm{loc}}$, $\forall a, b \in \mathbb{C}$, $\forall t\in\mathbb{R}$, the properties

$$\begin{aligned} \alpha_t(aA+bB) &= a\alpha_t(A) + b\alpha_t(B) \\ \alpha_t(AB) &= \alpha_t(A)\alpha_t(B) \\ \alpha_t(\mathbb{1}) &= \mathbb{1} \\ \alpha_t(A^*) &= \alpha_t(A)^*\,. \end{aligned}$$

(ii) For every $A \in \mathcal{A}_{\mathrm{loc}}$ and every $t\in\mathbb{R}$ define $\alpha_t^\Lambda(A) := e^{\mathrm{i}tH_\Lambda} A\, e^{-\mathrm{i}tH_\Lambda}$. Prove that, for sufficiently small $t\in\mathbb{R}$, one has

$$\lim_{\substack{\Lambda\to\infty \\ \Lambda\in\mathcal{F}(\mathbb{Z}^d)}} \|\alpha_t^\Lambda(A) - \alpha_t(A)\| \;=\; 0\,.$$

(iii) Prove that, for sufficiently small $t\in\mathbb{R}$, the map $\alpha_t : \mathcal{A}_{\mathrm{loc}} \to \mathcal{A}$ extends to a $*$-automorphism $\alpha_t : \mathcal{A} \to \mathcal{A}$.