HOMEWORK ASSIGNMENT – WEEK 05
Hand-in deadline: Thu 15 May by 12 p.m. in the "MSP" drop box.
Rules: Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/SS14_MSP.html`

Exercise 13. Consider the Weyl algebra $\mathcal{A}_{\mathrm{CCR}}(\mathbb{C})$, i.e., the CCR C^*-algebra where the underlying Hilbert space is the one-dimensional space $\mathbb{C}$. Next to the natural Fock representation $(\pi_{\mathrm{F}}, \mathscr{F}_+)$ of $\mathcal{A}_{\mathrm{CCR}}(\mathbb{C})$ consider the representation (the "*Schrödinger representation*")

$$\begin{aligned} \pi_{\mathrm{S}} \ &: \mathcal{A}_{\mathrm{CCR}}(\mathbb{C}) \to \mathcal{B}(L^2(\mathbb{R}, \mathrm{d}x)) \\ W(z) &\mapsto \pi_{\mathrm{S}}(W(z)) := e^{\frac{\mathrm{i}}{2}st} U(s)V(t) \\ z &= s + \mathrm{i}t, \quad s,t \in \mathbb{R}, \end{aligned}$$

where $\{U(s)\,|\,s \in \mathbb{R}\}$ and $\{V(t)\,|\,t \in \mathbb{R}\}$ are strongly continuous one-parameter unitary groups on $L^2(\mathbb{R}, \mathrm{d}x)$ defined on each $f \in L^2(\mathbb{R}, \mathrm{d}x)$ by

$$(U(s)f)(x) := e^{\mathrm{i}sx} f(x)\,, \qquad (V(t)f)(x) := f(x+t)\,, \qquad \text{for a.e. } x \in \mathbb{R}\,.$$

(i) Prove that the representation π_{S} is regular.

(ii) Prove that $L^2(\mathbb{R}, \mathrm{d}x)$ carries a Fock space structure given by the operators

$$a_{\mathrm{S}} \ := \ \frac{1}{\sqrt{2}}\Big(x + \frac{\mathrm{d}}{\mathrm{d}x}\Big)\,, \qquad a^*_{\mathrm{S}} \ := \ \frac{1}{\sqrt{2}}\Big(x - \frac{\mathrm{d}}{\mathrm{d}x}\Big)\,,$$

defined on the class of smooth and rapidly decreasing functions on $\mathbb{R}$, that is,

- prove that a_{S} and a^*_{S} satisfy the CCR,
- find the normalised function $\Omega_{\mathrm{S}} \in L^2(\mathbb{R}, \mathrm{d}x)$ that gives the vacuum w.r.t. a_{S},
- introduce the one-dimensional subspaces $\mathcal{H}_n \subset L^2(\mathbb{R}, \mathrm{d}x)$, $\mathcal{H}_n := \{\lambda (a^*_{\mathrm{S}})^n \Omega_{\mathrm{S}}\,|\,\lambda \in \mathbb{C}\}$, and prove that $L^2(\mathbb{R}, \mathrm{d}x) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$.
  (*Hint:* use well known properties of the quantum harmonic oscillator.)

(iii) Conclude that the Fock and the Schrödinger representations of $\mathcal{A}_{\mathrm{CCR}}(\mathbb{C})$ are unitarily equivalent via the unitary defined by $(a^*_{\mathrm{S}})^n \Omega_{\mathrm{S}} \mapsto (a^*_{\mathrm{F}})^n \Omega_{\mathrm{F}}$.

Exercise 14. Consider the Schrödinger representation $\pi_{\mathrm{S}} : \mathcal{A}_{\mathrm{CCR}}(\mathbb{C}) \to \mathcal{B}(L^2(\mathbb{R}, \mathrm{d}x))$ of the Weyl algebra $\mathcal{A}_{\mathrm{CCR}}(\mathbb{C})$ over $L^2(\mathbb{R}, \mathrm{d}x)$ introduced in Exercise 13. Prove that π_{S} is irreducible.

(*Hint:* if not, there would exist an invariant proper subspace $\mathcal{H}_1 \subset \mathcal{H}$, but imposing that $\phi \perp \mathcal{H}_1$ deduce that $\phi \equiv 0$.)

Exercise 15. (Coherent states in the bosonic Fock space)

Consider the bosonic Fock space $\mathfrak{F}_+(\mathfrak{h})$ over a given separable Hilbert space $\mathfrak{h}$. For computational convenience here, consider the "modified" Weyl operator $W(f) := e^{\overline{a^*(f)-a(f)}}$. (In practice, $W(f)$ is nothing but the Weyl operator defined in class associated with the function $\mathrm{i}f$ and apart from an irrelevant normalisation. This is only to spare you many extra pre-factors in the following.)

(i) Given $f \in \mathfrak{h}$, the state $W(f)\Omega \in \mathfrak{F}_+(\mathfrak{h})$ is called COHERENT STATE with one-particle state f. Prove that

$$W(f)\Omega \;=\; e^{-\|f\|^2/2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\, f^{\otimes n}\,,$$

where $f^{\otimes n}$ indicates the Fock-vector $\{0,\dots,f^{\otimes n},0,\dots\}$ and $\|\;\|$ is the norm in $\mathfrak{h}$. Interpret this result in terms of a Poisson distribution.

(ii) Prove that the expectation of the number of particles in the coherent state of f is $\|f\|^2$, namely prove that

$$\big\langle W(f)\Omega\,,\,\mathcal{N}\,W(f)\Omega\big\rangle_{\mathfrak{F}(\mathfrak{h})} \;=\; \|f\|^2 \;=\; \sum_{n=1}^{\infty}\big\langle W(f)\Omega, a^*(f_n)a(f_n)W(f)\Omega\big\rangle_{\mathfrak{F}(\mathfrak{h})}\,,$$

where, in the second identity, $(f_n)_{n=1}^{\infty}$ is an orthonormal basis of $\mathfrak{h}$.

(iii) Let $N \in \mathbb{N}$ and $f \in \mathfrak{h}$ with $\|f\| = 1$. Consider the factorised N-particle state

$$\Psi_N \;:=\; \{0,\dots,0,f^{\otimes N},0,0,\dots\} \;\in\; \mathfrak{F}_+(\mathfrak{h})\,.$$

Prove that Ψ_N can be expressed as the following linear superposition of coherent states:

$$\Psi_N \;=\; C_N \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\, e^{i\theta N}\, W(e^{-\mathrm{i}\theta}\sqrt{N}f)\Omega$$

with the constant $C_N := \dfrac{\sqrt{N!}}{N^{N/2}e^{-N/2}}$. (Note that $C_N \sim N^{1/4}$ as $N \to \infty$.)

Exercise 16. (In this exercise do the necessary algebraic manipulations in a formal way, namely with no concern with domain issues of unbounded operators.)

Consider:

- the Weyl algebra $\mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})$ over a given separable Hilbert space $\mathfrak{h}$,
- a real linear invertible map $S : \mathfrak{h} \to \mathfrak{h}$ such that $\mathfrak{Im}\langle Sf, Sg\rangle \;=\; \mathfrak{Im}\langle f, g\rangle\;\forall f, g \in \mathfrak{h}$, and the corresponding Bogoliubov $*$-automorphism $\gamma : \mathcal{A}_{\mathrm{CCR}}(\mathfrak{h}) \to \mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})$ defined by $\gamma(W(f)) := W(Sf)\;\forall f \in \mathfrak{h}$,
- the Fock representation $(\pi, \mathfrak{F}_+)$ of $\mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})$ (recall that this representation is *regular*).

Correspondingly, for every $f \in \mathfrak{h}$ let $\widetilde{\Phi}_\pi(f)$, $\widetilde{a}_\pi(f)$, and $\widetilde{a}^*_\pi(f)$ be the (unbounded) operators on $\mathfrak{F}_+$ defined respectively by

$$\begin{aligned} e^{\mathrm{i}\,\widetilde{\Phi}_\pi(f)} &:= \pi(\gamma(W(f))) \\ \widetilde{a}_\pi(f) &:= \frac{1}{\sqrt{2}}\big(\widetilde{\Phi}_\pi(f) + \mathrm{i}\widetilde{\Phi}_\pi(\mathrm{i}f)\big) \\ \widetilde{a}^*_\pi(f) &:= \frac{1}{\sqrt{2}}\big(\widetilde{\Phi}_\pi(f) - \mathrm{i}\widetilde{\Phi}_\pi(\mathrm{i}f)\big)\,. \end{aligned}$$

(i) Find in terms of S two maps $L : \mathfrak{h} \to \mathfrak{h}$ and $A : \mathfrak{h} \to \mathfrak{h}$ such that, for every $f \in \mathfrak{h}$,

$$\begin{aligned} \widetilde{a}_\pi(f) &= a(Lf) + a^*(Af) \\ \widetilde{a}^*_\pi(f) &= a(Af) + a^*(Lf)\,. \end{aligned}$$

Are these maps complex linear?

(*Hint:* use the fact that $\pi : \mathcal{A}_{\mathrm{CCR}}(\mathfrak{h}) \to \mathcal{B}(\mathfrak{F}_+)$ is regular to give meaning to the generator of $t \mapsto \pi(W(tf))$, $t \in \mathbb{R}$, and compare it with $\widetilde{\Phi}_\pi(f)$.)

(ii) Prove that imposing that $\widetilde{a}_\pi(f)$ and $\widetilde{a}^*_\pi(g)$ satisfy the CCR as operators on $\mathfrak{F}_+$ yields

$$\begin{aligned} L^*L - A^*A &= \mathbb{1} = LL^* - AA^* \\ L^*A - A^*L &= 0 = AL^* - LA^*\,. \end{aligned}$$

(*Hint:* derive two of the four identities above directly from the CCR; as for the other two, observe that $f \mapsto e^{\mathrm{i}\,\widetilde{\Phi}_\pi(f)}$ and $f \mapsto e^{\frac{\mathrm{i}}{\sqrt{2}}\overline{(a(f)+a^*(f))}}$ are two Weyl operators on the same CCR algebra and apply a uniqueness result stated in class: the invertibility of the correspondence $a(f)$,$a^*(g) \leftrightarrow \widetilde{a}_\pi(f)$, $\widetilde{a}^*_\pi(g)$ yields the other two identities requested above.)

(iii) Assume that $\mathrm{Tr}A^*A < \infty$. Prove that

$$\langle \Omega, \widetilde{\mathcal{N}}_\pi \Omega \rangle_{\mathfrak{F}} = \mathrm{Tr}A^*A\,,$$

where $\widetilde{\mathcal{N}}_\pi := \sum_{n=1}^\infty \widetilde{a}^*_\pi(f_n)\widetilde{a}_\pi(f_n))$ and $(f_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathfrak{h}$.