HOMEWORK ASSIGNMENT – WEEK 03
Hand-in deadline: Fri 2 May by 12 p.m. in the "MSP" drop box.
Rules: Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/SS14_MSP.html`

Exercise 5. Let $\mathcal{A}$ be a unital Banach algebra. Assume that every element $A \in \mathcal{A}$ satisfies the identity $\|A^2\| = \|A\|^2$.

(i) Let $A \in \mathcal{A}$ be such that $\|A^2\| = \|A\|^2$. Prove that the spectral radius $r(A) := \sup_{\lambda\in\sigma(A)} |\lambda|$ is such that $r(A) = \|A\|$.

(ii) Let $z \in \mathbb{C}$ and $A, B \in \mathcal{A}$, arbitrary. Prove that $\sigma(e^{-zA}Be^{zA}) = \sigma(B)$.
(The element $e^{zA} \in \mathcal{A}$ is defined by its norm-convergent series.)

(iii) Let $A, B \in \mathcal{A}$, arbitrary. Prove that $\|e^{-zA}Be^{zA}\|$ is constant for every $z \in \mathbb{C}$.

(iv) Deduce from (i)-(iii) above that the algebra $\mathcal{A}$ is necessarily commutative.
(*Hint:* $\mathbb{C} \ni z \mapsto e^{-zA}Be^{zA}$ is holomorphic.)

The purpose of the next two exercises is to compute the GNS representation of a C^*-algebra in two explicit cases. In one case $\mathcal{A}$ is commutative, in the other one $\mathcal{A}$ is not.

Exercise 6. Consider the C^*-algebra $C([0,1])$ (with the $\|\ \|_{\sup}$ norm). For each $f \in C([0,1])$ define $\omega(f) := \int_0^1 f(t)\mathrm{d}t$.

(i) Prove that ω is a state on $C([0,1])$. Is ω a pure state?

(ii) Find the GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ associated with the state ω, i.e., describe explicitly the Hilbert space $\mathcal{H}_\omega$, the $*$-homomorphism $\pi_\omega : C([0,1]) \to \mathcal{L}(\mathcal{H}_\omega)$, and the cyclic vector $\Omega_\omega \in \mathcal{H}_\omega$.

(iii) Prove that π_ω is a faithful representation, i.e., $\|\pi_\omega(f)\| = \|f\|_{\sup} \ \ \forall f \in C([0,1])$.

Exercise 7. Consider the C^*-algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$ of bounded linear operators on a given Hilbert space $\mathcal{H}$, and the (normal) state ω on $\mathcal{A}$ realised by the density matrix ρ, i.e., $\rho(A) = \mathrm{Tr}_{\mathcal{H}}(\rho A)$ $\forall A \in \mathcal{A}$, where $\rho : \mathcal{H} \to \mathcal{H}$ is bounded, self-adjoint, positive, and with $\mathrm{Tr}_{\mathcal{H}}\rho = 1$. Moreover, consider the triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ defined as follows:

- $\mathcal{H}_\omega := \mathcal{L}^2(\mathcal{K}, \mathcal{H}) \equiv \{$the Hilbert-Schmidt operators $\mathcal{K} \to \mathcal{H}\}$ where $\mathcal{K} := \overline{\operatorname{Ran} \rho}$ (thus, $\mathcal{K}$ is a Hilbert sub-space of $\mathcal{H}$). Recall that $T \in \mathcal{L}^2(\mathcal{K}, \mathcal{H})$ means that $T : \mathcal{K} \to \mathcal{H}$ is linear, bounded, and $\operatorname{Tr}_{\mathcal{K}}(T^*T) < \infty$. Equip $\mathcal{H}_\omega$ with the scalar product $\langle T, S\rangle_{\mathcal{H}_\omega} := \operatorname{Tr}_{\mathcal{K}}(T^*S)$ $\forall\, T, S \in \mathcal{H}_\omega$ which makes it, as well known, a Hilbert space.
- $\Omega_\omega := \rho^{1/2} \circ \imath$ where $\imath : \mathcal{K} \to \mathcal{H}$ is the canonical injection $\imath(x) = x$ $\forall x \in \mathcal{K}$.
- for each $A \in \mathcal{A}$ $\pi_\omega(A) : \mathcal{H}_\omega \to \mathcal{H}_\omega$, $\pi_\omega(A)T := AT$ $\forall\, T \in \mathcal{H}_\omega$.

(i) Prove that Ω_ω is a unit vector in $\mathcal{H}_\omega$.

(ii) Prove that π_ω is a faithful representation of the C^*-algebra $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_\omega)$.

(iii) Prove that Ω_ω is a cyclic vector for the representation π_ω.

(iv) Prove that $\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega\rangle_{\mathcal{H}_\omega}$ $\forall A \in \mathcal{A}$.

Exercise 8. Let $\mathcal{A}$ be a C^*-algebra with unit, ω be a state on $\mathcal{A}$, $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ be the corresponding GNS representation of $\mathcal{A}$.

(i) Assume that $(\mathcal{H}, \pi, \Omega)$ is another cyclic representation of $\mathcal{A}$ such that

$$\omega(A) = \langle \Omega, \pi(A)\Omega\rangle_{\mathcal{H}} \qquad \forall A \in \mathcal{A} .$$

Produce a unitary operator $U : \mathcal{H} \xrightarrow{\cong} \mathcal{H}_\omega$ (i.e., unitary from $\mathcal{H}$ onto $\mathcal{H}_\omega$) such that

$$\begin{aligned} \pi_\omega(A) &= U\pi(A)U^{-1} \qquad \forall A \in \mathcal{A} , \\ \Omega_\omega &= U\Omega . \end{aligned}$$

(ii) Assume that a one-parameter weakly continuous group $\{\alpha_t \,|\, t \in \mathbb{R}\}$ of $*$-automorphisms of $\mathcal{A}$ is given. Recall that this means that

- for each $t \in \mathbb{R}$ α_t is a $*$-automorphism on $\mathcal{A}$,
- $\forall\, t, s \in \mathbb{R}$: $\alpha_0 = \imath$ (the identity map over $\mathcal{A}$), $\alpha_t\alpha_s = \alpha_{t+s}$, $\alpha_t^{-1} = \alpha_{-t}$,
- for every state ρ on $\mathcal{A}$ and every $A \in \mathcal{A}$, $\rho(\alpha_t(A)) \xrightarrow{t \to 0} \rho(A)$.

Assume now that there is a state ω on $\mathcal{A}$ which is invariant with respect to α_t, namely such that

$$\omega(\alpha_t(A)) = \omega(A) \qquad \forall A \in \mathcal{A} , \ \ \forall\, t \in \mathbb{R} ,$$

and denote by $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ the corresponding GNS representation. Prove that there exists a densely defined self-adjoint operator H on the Hilbert space $\mathcal{H}_\omega$ such that

$$\begin{aligned} \pi_\omega(\alpha_t(A)) &= e^{\mathrm{i}tH}\pi_\omega(A)\, e^{-\mathrm{i}tH} \qquad \forall A \in \mathcal{A} \quad \forall\, t \in \mathbb{R}. \\ \Omega_\omega \in \text{domain of } H \text{ and } H\Omega_\omega &= 0 . \end{aligned}$$