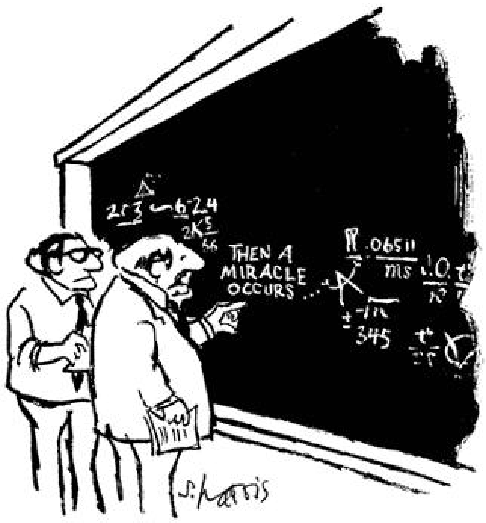
Mathematical Statistical Physics

TMP Programme Munich – spring term 2014

HOMEWORK ASSIGNMENT – WEEK 02
Hand-in deadline: Thursday 24 April 2014 by 12 p.m. in the "MSP" drop box.
Rules: Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/SS14_MSP.html`



"I think you should be more explicit here in step two."

Exercise 1. In each of the following cases decide whether the set $\mathcal{A}$ equipped with the structure declared below is a C^*-algebra (justify your answer).

(i) $\mathcal{A} = \mathbb{C}^n$ (for some $n \in \mathbb{N}$), with component-wise sum, product, and complex conjugation, equipped with the p-norm

$$\|\mathbf{x}\| \;:=\; \begin{cases} \Big(\sum_{i=1}^n |x_i|^p \Big)^{1/p} & \text{if } 1 \leqslant p < \infty \\ \max_i |x_i| & \text{if } p = \infty \end{cases}$$

where $\mathbf{x} = (x_1, \dots, x_n)$.

(ii) $\mathcal{A} = \mathcal{M}(n \times n, \mathbb{C})$, the $*$-algebra of $n \times n$ complex matrices ($n \in \mathbb{N}$), equipped with the norm

$$\|A\|_\bullet^2 \;:=\; \mathrm{Tr}(A^*A) \;=\; \sum_{i,j=1}^n |a_{ij}|^2$$

where $A = (a_{ij})$.

(iii) $\mathcal{A} = C^k([0,1])$, the $*$-algebra of k-times differentiable functions on $[0,1]$ with continuous k-th derivative (for some $k \in \{0,1,2,3,\dots\}$), equipped with the supremum norm $\|\;\|_{\sup}$.

(iv) $\mathcal{A} = \mathcal{J}_1(\mathcal{H})$, the $*$-subalgebra of bounded operators A on a Hilbert space $\mathcal{H}$ such that $\mathrm{Tr}|A| < \infty$, equipped with the inherited operator norm.

Exercise 2. Let $\lambda \in \mathbb{R}$. Consider the matrix

$$A = \begin{pmatrix} 1 - 3\cos 2\lambda & 3\,\mathrm{i}\sin 2\lambda & 2\,\mathrm{i}\sin\lambda \\ -3\,\mathrm{i}\sin 2\lambda & 1 + 3\cos 2\lambda & 2\cos\lambda \\ 0 & 0 & 4 \end{pmatrix}$$

as an element in the C^*-algebra $\mathcal{A} = \mathcal{M}(3 \times 3, \mathbb{C})$ of 3×3 complex matrices.

(i) Find the C^*-subalgebra of $\mathcal{A}$ generated by A, A^*, and the unit matrix.
(*Hint:* exploit a convenient basis.)

(ii) Is the C^*-algebra found in (ii) commutative?

Exercise 3. Consider the C^*-algebra $\mathcal{A} = C([-1,1])$ of the complex-valued continuous functions over $[-1,1]$, with the usual point-wise sum, product, and complex conjugation, and with the supremum norm. Let E be a closed subset of $[-1,1]$. Set

$$\begin{aligned} \mathcal{I} &:= \{ f \in \mathcal{A} \,|\, f(x) = 0 \;\forall x \in E \} \\ \mathcal{J} &:= \{ f \in \mathcal{A} \,|\, f = xg \text{ for some } g \in \mathcal{A} \}\,. \end{aligned}$$

(i) Prove that $\mathcal{I}$ is a two-sided closed $*$-ideal of $\mathcal{A}$.

(ii) Prove that the quotient algebra $\mathcal{A}/\mathcal{I}$ is identifiable as $C(E)$.

(iii) Prove that $\mathcal{J}$ is a two-sided $*$-ideal of $\mathcal{A}$.

(iv) Is $\mathcal{J}$ closed? Give a proof of its closedness or find its closure $\overline{\mathcal{J}}$ in $\mathcal{A}$.

Exercise 4. Let $\mathcal{A}$ be a C^*-algebra with unit and denote by $G(\mathcal{A})$ the group of all invertible elements in $\mathcal{A}$.

(i) Prove that the group $G(\mathcal{A})$ is an open subset of $\mathcal{A}$. More precisely, prove that if $\|A - A_0\| < 1/\|A_0^{-1}\|$ for an $A_0 \in G(A)$ then A is invertible and

$$A^{-1} = \Big(\sum_{n=0}^{\infty} \big(A_0^{-1}(A_0 - A) \big)^n \Big) A_0^{-1}\,.$$

(ii) Prove that the map $A \mapsto A^{-1}$ is a continuous map on $G(\mathcal{A})$.

(iii) How do the answers to (i) and (ii) change if $\mathcal{A}$ is only assumed to be a Banach algebra?