LMU | LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN | MATHEMATISCHES INSTITUT

Spring term 2013 / *Sommersemester 2013*

Mathematical Statistical Physics – Final exam, 19.7.2013

Mathematische Statistische Physik – Endklausur, 19.7.2013

Name:/*Name*: ______________________ **Pseudonym:**/*Pseudonym:* __________

Matriculation number:/*Matrikelnr.:* __________ **Semester:**/*Fachsemester:* __________

Degree course:/*Studiengang:* ❏ Diplom ❏ Bachelor, PO ______ ❏ Lehramt Gymnasium (modularisiert)
❏ TMP ❏ Master, PO ______ ❏ Lehramt Gymnasium (nicht modul.)
❏ ______________________

Major:/*Hauptfach:* ❏ Mathematik ❏ Wirtschaftsm. ❏ Informatik ❏ Physik ❏ Statistik ❏ ______

Minor:/*Nebenfach:* ❏ Mathematik ❏ Wirtschaftsm. ❏ Informatik ❏ Physik ❏ Statistik ❏ ______

Credits needed for:/*Anrechnung der Credit Points für das:* ❏ Hauptfach ❏ Nebenfach

Extra solution sheets submitted:/*Zusätzlich abgegebene Lösungsblätter:* ❏ Yes ❏ No

problem	**1**	**2**	**3**	**4**	**5**	**6**	$\sum$
total marks	**10**	**10**	**10**	**10**	**10**	**10**	60
scored marks							

homework bonus		**final test performance**		**total performance**		**FINAL MARK**	

INSTRUCTIONS:

- This booklet is made of sixteen pages, including the cover, numbered from 1 to 16. The test consists of six problems. Each problem is worth 10 marks. 50 marks are counted as 100% performance in this test. You are free to work on any problem and to collect partial credits.
- The only material that you are allowed to use is black or blue pens/pencils and one hand-written, two-sided, A4-size "cheat sheet" (Spickzettel).
- Raise up your hand to request extra sheets or scratch paper. You are not allowed to use your own paper.
- Prove all your statements or refer to the standard material discussed in class.
- Work individually. Write with legible handwriting. You may hand in your solution in German or in English. Put your name on every sheet you hand in.
- You have 180 minutes.

GOOD LUCK!

Fill in the form here below only if you need the certificate (Schein).

Dieser Leistungsnachweis entspricht auch den Anforderungen				
nach §	Abs.	Nr.	Buchstabe	LPO I
nach §	Abs.	Nr.	Buchstabe	LPO I

UNIVERSITÄT MÜNCHEN

ZEUGNIS

Der / Die Studierende der ____________________

Herr / Frau ________________ *aus* ________________

geboren am ________ *in* ____________ *hat im* **SoSe** *–Halbjahr* **2013**

meine Übungen **zur Mathematischen Statistischen Physik**

mit ____________________ *besucht.*

Er / Sie hat ____________________

schriftliche Arbeiten geliefert, die mit ihm / ihr besprochen wurden.

MÜNCHEN, *den* **19 Juli 2013**

Name

PROBLEM 1. (10 marks)

Let $\mathcal{A}$ be a unital C^*-algebra and let $A \in \mathcal{A}$ be a normal element, i.e., $[A, A^*] = \mathbb{O}$. Let $B \in \mathcal{A}$ be such that $[B, A] = \mathbb{O}$.

(i) For each $z \in \mathbb{C}$ define $U(z) := e^{zA^* - \overline{z}A}$. Prove that $U(z)$ is a unitary element of $\mathcal{A}$.

(ii) Let ω be an arbitrary state on $\mathcal{A}$. Prove that the function $z \mapsto F(z) := \omega(U(-z)BU(z))$ is constant over $\mathbb{C}$.

(*Hint:* Liouville's theorem.)

(iii) Prove that $[B, A^*] = \mathbb{O}$.

SOLUTION:

SOLUTION TO PROBLEM 1 (CONTINUATION):

Name

PROBLEM 2. (10 marks)

(i) Let $\mathcal{A}$ be a unital C^*-algebra and let $A, B \in \mathcal{A}$ be such that $[A, B] = \mathbb{O}$. Prove that

$$\mathbb{O} \leqslant A \leqslant B \quad \Rightarrow \quad \mathbb{O} \leqslant A^2 \leqslant B^2 \,.$$

(*Hint:* consider the C^*-algebra generated by A and B. It is commutative (why?).)

(ii) Consider the case $\mathcal{A} = \mathcal{M}_{2\times 2}(\mathbb{C})$ and matrices of the form $A = \left(\begin{smallmatrix} a & b \\ \bar{b} & 1 \end{smallmatrix}\right)$, $a \in \mathbb{R}$, $b \in \mathbb{C}$. Find a condition on a, b which is equivalent to $A \geqslant \mathbb{O}$.

(iii) In the case $\mathcal{A} = \mathcal{M}_{2\times 2}(\mathbb{C})$ give an example of

$$\left.\begin{aligned} A, B &\in \mathcal{A} \\ A &\geqslant \mathbb{O} \\ B &\geqslant \mathbb{O} \end{aligned}\right\} \quad \text{but} \quad AB \not\geqslant \mathbb{O} \,.$$

SOLUTION:

SOLUTION TO PROBLEM 2 (CONTINUATION):

Name

PROBLEM 3. (10 marks)

Let $d \in \mathbb{N}$. Consider the CCR algebra $\mathcal{A} = \mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})$ over the Hilbert space $\mathfrak{h} := L^2(\mathbb{R}^d, \mathbb{C})$. Denote as usual by $W(f)$, $f \in \mathfrak{h}$, the Weyl operators generating $\mathcal{A}$. (Recall the definition: $W(f) := \frac{1}{\sqrt{2}} e^{\mathrm{i}\overline{(a^*(f)+a(f))}}$.)

(i) For every $v \in \mathbb{R}^d$ and every $f \in L^2(\mathbb{R}^d, \mathbb{C})$ define $(U_v f)(x) := f(x-v)$ for a.e. $x \in \mathbb{R}^d$ and, correspondingly, define the Bogoliubov transformation

$$\tau_v(W(f)) \;:=\; W(U_v f)\,,$$

extended as usual by linearity on the whole $\mathrm{Span}\{W(f) \,|\, f \in \mathfrak{h}\}$. Prove that $\{\tau_v\}_{v\in\mathbb{R}^d}$ extends to a $\mathbb{R}^d$-parameter group of $*$-automorphisms of $\mathcal{A}$.

(ii) Prove that

$$\lim_{\substack{v\in\mathbb{R}^d\\ |v|\to\infty}} \big\|\,[\tau_v(W(f)), W(g)]\,\big\| \;=\; 0$$

for any $f, g \in L^2(\mathbb{R}^d)$.

(*Hint:* compute/estimate $\big\|\,[\tau_v(W(f)), W(g)]\,\big\|$ using the properties of the Weyl operator W; then prove and use the limit

$$\lim_{\substack{v\in\mathbb{R}^d\\ |v|\to\infty}} \int_{\mathbb{R}^d} \phi(x-v)\,\psi(x)\,\mathrm{d}x \;=\; 0$$

valid for any $\phi, \psi \in L^2(\mathbb{R}^d, \mathbb{C})$.)

(iii) Prove that $\{\tau_v\}_{v\in\mathbb{R}^d}$ is asymptotically abelian in the norm sense, namely

$$\lim_{\substack{v\in\mathbb{R}^d\\ |v|\to\infty}} \big\|\,[\tau_v(A), B]\,\big\| \;=\; 0$$

for any $A, B \in \mathcal{A}$.

SOLUTION:

SOLUTION TO PROBLEM 3 (CONTINUATION):

Name

PROBLEM 4. (10 marks)

Consider the CAR algebra $\mathcal{A} = \mathcal{A}_{\mathrm{CAR}}(\mathfrak{h})$ over a given Hilbert space $\mathfrak{h}$. Denote as usual by $a^*(f)$ and $a(f)$, $f \in \mathfrak{h}$, respectively the creation and the annihilation operators on $\mathcal{A}$. Let H be a positive Hamiltonian on $\mathfrak{h}$. Consider the group $\{\tau_t\}_{t\in\mathbb{R}}$ of Bogoliubov transformations on $\mathcal{A}$ defined by

$$\tau_t(a^*(f)) \;:=\; a^*(e^{\mathrm{i}tH}f)\,, \qquad \tau_t(a(f)) \;:=\; a(e^{\mathrm{i}tH}f)\,, \qquad f \in \mathfrak{h}\,, \tag{$\bullet$}$$

extended as usual by linearity and density on the whole $\mathcal{A}$. Correspondingly, and for some $\beta > 0$, let ω be a (τ_t, β)-KMS state over $\mathcal{A}$.

Compute the two-point functions on $\mathcal{A}$ associated with ω, namely compute the quantity

$$\omega(a^*(f)a(g))$$

for any $f, g \in \mathfrak{h}$.

(*Hint:* in class $\omega(a^*(f)a(g))$ was computed under the assumption that ω is a Gibbs state and that $e^{-\beta H}$ is of trace class. Note that here you are asked to re-do the computation for a case where a priori ω is *not* a Gibbs state and $e^{-\beta H}$ is *not* of trace class. Instead, you are supposed to use

- the KMS condition satisfied by ω,
- the CARs,
- and the properties ($\bullet$).)

Note: although the KMS condition involves a suitable *dense* and *τ-invariant* $*$-subalgebra of $\mathcal{A}$, proceed in the solution to this problem considering only the $f, g \in \mathfrak{h}$ for which the KMS condition can be applied (the extension to any f, g is a density argument that you are not asked to perform in this solution).

SOLUTION:

SOLUTION TO PROBLEM 4 (CONTINUATION):

Name

PROBLEM 5. (10 marks)

Given a Hilbert space $\mathfrak{h}$, with norm $\| \ \|$, consider the bosonic Fock space $\mathfrak{F}_+(\mathfrak{h})$ with the usual notation for the annihilation, creation,and number operator, respectively $a(f)$, $a^*(f)$, and $\mathcal{N}$, where $f \in \mathfrak{h}$. Consider the (modified) Weyl operators $W(f) := e^{\overline{a^*(f)-a(f)}}$, $f \in \mathfrak{h}$. Denote by Ω the vacuum in the Fock space.

(i) Prove that

$$W(f)\Omega \;=\; e^{-\|f\|^2/2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\, f^{\otimes n}\,,$$

where $f^{\otimes n}$ indicates the Fock-vector $\{0, \dots, f^{\otimes n}, 0, \dots\}$.

(ii) Prove that the expectation of the number of particles in the state $W(f)\Omega$ is $\|f\|^2$, namely prove that

$$\big\langle W(f)\Omega\,,\,\mathcal{N}\,W(f)\Omega\big\rangle_{\mathfrak{F}(\mathfrak{h})} \;=\; \|f\|^2 \;=\; \sum_{n=1}^{\infty} \big\langle W(f)\Omega, a^*(f_n)a(f_n)W(f)\Omega\big\rangle_{\mathfrak{F}(\mathfrak{h})}\,,$$

where the second identity is understood under the additional assumption that $\{f_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathfrak{h}$.

SOLUTION:

SOLUTION TO PROBLEM 5 (CONTINUATION):

Name

PROBLEM 6. (10 marks) – SHORT QUESTIONS

Answer to each question with a YES or a NO. Marking scheme: 1 mark for each correct answer, 0 marks for each unanswered question, -1 mark for each wrong answer.

6.1 Is there a unique KMS state for an infinite block of iron at room temperature?

❏ YES ❏ NO

6.2 Consider the C^*-algebra $\mathcal{A}$ of 17×17 complex-valued matrices. Is $\omega(A) = \frac{1}{17}\sum_{i,j=1}^{17} A_{ij}$, $A = (A_{ij}) \in \mathcal{A}$, a state over $\mathcal{A}$?

❏ YES ❏ NO

6.3 Equip the C^*-algebra $\mathcal{A}$ considered in Question 6.2 with a new norm $\|A\|_\sim := (\mathrm{Tr}(A^*A))^{1/2}$. Does this new norm turn $\mathcal{A}$ into a C^*-algebra?

❏ YES ❏ NO

6.4 Consider the C^*-algebra $\mathcal{A} = \mathcal{L}(L^2(\mathbb{R}))$ and the time evolution $\tau_t(A) = e^{\mathrm{i}tH} A e^{-\mathrm{i}tH}$, $t \in \mathbb{R}$, where H is the Hamiltonian of the 1D harmonic oscillator, namely $(Hf)(x) := -f''(x) + x^2 f(x)$ on the domain $\mathcal{D} = \{f \in L^2(\mathbb{R}) \,|\, -f'' + x^2 f \in L^2(\mathbb{R})\}$. Is τ_t asymptotically abelian?

❏ YES ❏ NO

6.5 With the usual meaning of the symbols as from class and homework, is it correct that the group of Bogoliubov transformations $\tau_t(W(f)) = W(e^{\mathrm{i}tH} f)$ is strongly continuous for the free Fermi gas and not strongly continuous for the free Bose gas?

❏ YES ❏ NO

6.6 Consider the C^*-algebra of 2×2 matrices over $\mathbb{C}$ and the state $\omega(A) = \mathrm{Tr}(\rho A)$ where $\rho = \frac{1}{3}|+\rangle\langle+| + \frac{2}{3}|-\rangle\langle-|$, with the notation $|+\rangle = \binom{1}{0}$ and $|-\rangle = \binom{0}{1}$. Is ω a primary state?

❑ YES ❑ NO

6.7 Given a C^*-algebra $\mathcal{A}$, a dynamics $\{\tau_t\}_{t\in\mathbb{R}}$ on $\mathcal{A}$, and an inverse temperature $\beta \in \mathbb{R}$, is it necessarily true that a (τ_t, β)-KMS state is stationary in time?

❑ YES ❑ NO

6.8 With the usual meaning of the symbols as from class and homework, consider the 1D Ising model at non-zero temperature T and with magnetic field B. Is it true that at the Renormalisation Group limit point one has $\langle S_1 S_2 \rangle = \langle S_1 \rangle \langle S_2 \rangle$?

❑ YES ❑ NO

6.9 With the usual meaning of the symbols as from class and homework, consider the 2D Ising model at non-zero temperature T and with magnetic field B. Is it true that $\langle S_1 S_2 \rangle = \langle S_1 \rangle \langle S_2 \rangle$?

❑ YES ❑ NO

6.10 With the usual meaning of the symbols as from class and homework, consider the 3D Ising model. If you plot the logarithm of the spontaneous magnetization as a function of $\log \frac{(T-T_c)}{T_c}$ for T approaching T_c from below, do you get a straight line?

❑ YES ❑ NO