ADDITIONAL PROBLEMS, WEEK 05

Note: Additional problems are handed out every week as a supplement to the material discussed in class and in the homework. They are not part of the homework load, no solution has to be submitted, they bring no credit for the final mark. On the other hand, they are meant to provide further examples and applications to the notions presented in class, as well as further material for your own preparation at home for the final test.

Info: `www.math.lmu.de/~michel/SS13_MSP.html`

Problem 19. Let $\mathcal{A}$ be a C^*-algebra, $\{\alpha_t\}_{t\in\mathbb{R}}$ be a one-parameter strongly continuous group of $*$-automorphisms of $\mathcal{A}$ (i.e., for any $A \in \mathcal{A}$, $t \to 0 \Rightarrow \|\alpha_t(A) - A\| \to 0$), and ω be an α_t-invariant state on $\mathcal{A}$, namely $\omega(\alpha_t(A)) = \omega(A)\ \forall A \in \mathcal{A}$. Consider the GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of $\mathcal{A}$ associated with ω.

(i) Prove that for every $t \in \mathbb{R}$ there exists a unitary operator U_t acting on $\mathcal{H}_\omega$ such that

$$\begin{aligned} \pi_\omega(\alpha_t(A)) &= U_t \pi_\omega(A) U_t^* \qquad \forall A \in \mathcal{A}\,, \\ U_t \Omega_\omega &= \Omega_\omega\,. \end{aligned}$$

(*Hint:* define U_t first on the dense $\pi_\omega(\mathcal{A})$ as $U_t\pi_\omega(A)\Omega_\omega := \pi_\omega(\alpha_t(A))\Omega_\omega$, prove its boundedness, extend by density.)

(ii) Prove that $\{U_t\}_{t\in\mathbb{R}}$ is one-parameter strongly continuous group of unitary operators acting on $\mathcal{H}_\omega$.

(iii) Prove that there exists a self-adjoint operator H acting on $\mathcal{H}_\omega$ such that $U_t = e^{\mathrm{i}tH}$ for all $t \in \mathbb{R}$ and $H\Omega_\omega = 0$.

Problem 20. Let $\mathcal{A}$ be a unital C^*-algebra, $\{\alpha_t\}_{t\in\mathbb{R}}$ be a one-parameter continuous group of $*$-automorphisms of $\mathcal{A}$, and ω be an α_t-invariant state on $\mathcal{A}$.

(i) Prove that there exists a dense $*$-subalgebra D_δ of $\mathcal{A}$ such that

$$\exists \lim_{t\to 0} \frac{\alpha_t(A) - A}{t} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\,\alpha_t(A)\Big|_{t=0} \;=:\; \delta(A)\,.$$

(ii) Prove that $\delta : D_\delta \to \mathcal{A}$, called the INFINITESIMAL GENERATOR OF α_t, is a linear map such that, for any $A, B \in D_\delta$,

$$\begin{aligned} \delta(AB) &= \delta(A)B + A\delta(B)\,, \\ \delta(A^*) &= \delta(A)^*\,, \end{aligned}$$

and δ is closable.

(iii) Prove that $(\mathbb{1} + a\delta)D_\delta$ is dense in $\mathcal{A}$ for any $a \in \mathbb{R} \setminus \{0\}$.

(iv) Prove that $\omega(\delta(A)) = 0$ for all $A \in \mathcal{A}$

Problem 21. Let $\mathcal{A}$ be a C^*-algebra, $\{\alpha_t\}_{t\in\mathbb{R}}$ be a one-parameter strongly continuous group of $*$-automorphisms of $\mathcal{A}$, and ω be an α_t-invariant state on $\mathcal{A}$. Let δ be the infinitesimal generator of α_t and let H be the self-adjoint generator of the dynamics in the GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ of $\mathcal{A}$ associated with ω (Problem 19), in particular $\pi_\omega(\alpha_t(A)) = e^{\mathrm{i}tH}\pi_\omega(A)e^{-\mathrm{i}tH}$ $\forall t \in \mathbb{R}$, $\forall A \in \mathcal{A}$. Denote by $\mathcal{D}(H)$ the domain of H in $\mathcal{H}_\omega$.

(i) Let $A \in \mathcal{A}$. Prove that on the subspace $V := \{\psi \,|\, \psi \in \mathcal{D}(H), \pi_\omega(A)\psi \in \mathcal{D}(H)\}$ one has

$$\pi_\omega(\delta(A))\,\psi \;=\; [\,\mathrm{i}H\,,\,\pi_\omega(A)\,]\,\psi \qquad \forall \psi \in V.$$

(ii) Prove that $\pi_\omega(D_\delta)\Omega_\omega \subset D(H)$ and

$$\pi_\omega(\delta(A))\phi \;=\; [\,\mathrm{i}H\,,\,\pi_\omega(A)\,]\,\phi \qquad \forall \phi \in \pi_\omega(D_\delta)\Omega_\omega \quad \forall A \in D_\delta\,.$$

(iii) Assume further that $t \mapsto \omega(\alpha_t(A))$ is differentiable for some $A \in \mathcal{A}$. Prove that

$$\omega(\delta(A)) \;=\; \langle \Omega_\omega, [\,\mathrm{i}H\,,\,\pi_\omega(A)\,]\,\Omega_\omega\rangle\,.$$

Problem 22. Let $\mathcal{A} = \bigcup_{n=1}^\infty \mathcal{A}_n$ be a UHF (uniformly hyper-finite) quasi-local algebra.

(i) Let $D_\delta := \bigcup_{n=1}^\infty \mathcal{A}_n$ and let $\delta : D_\delta \to \mathcal{A}$ be a linear map such that, for any $A, B \in D_\delta$,

$$\begin{aligned} \delta(AB) \;&=\; \delta(A)B + A\delta(B)\,,\\ \delta(A^*) \;&=\; \delta(A)^*\,. \end{aligned}$$

Prove that δ is closable and that there exist elements $H_n = H_n^* \in \mathcal{A}_n$, $n \in \mathbb{N}$, such that

$$\delta(A) \;=\; \delta_n(A) \;:=\; [\mathrm{i}H_n, A] \qquad \forall A \in \mathcal{A}_n\,.$$

(ii) Prove that if $(\mathbb{1} \pm \delta)\big(\bigcup_{n=1}^\infty \mathcal{A}_n\big)$ are dense in $\mathcal{A}$ then the closure $\overline{\delta}$ of δ is the generator of a strongly continuous one-parameter group $\{\alpha_t\}_{t\in\mathbb{R}}$ of $*$-automorphisms of $\mathcal{A}$, and moreover

$$\alpha_t(A) \;=\; \lim_{n\to\infty} e^{t\delta_n}(A) \qquad \forall A \in \mathcal{A}$$

where the limit exists in norm, uniformly for t in finite intervals.

Problem 23. Let ω be a state on a C^*-algebra $\mathcal{A}$ and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a one-parameter continuous group of $*$-automorphisms of $\mathcal{A}$.

(i) For each $T > 0$ prove that $\mathcal{A} \ni A \mapsto \omega_T(A) := \frac{1}{T}\int_0^T \mathrm{d}t\,\omega(\alpha_t(A))$ defines a state on $\mathcal{A}$.

(ii) Out of the family $\{\omega_T \,|\, T > 0\}$ construct an α_t-invariant state on $\mathcal{A}$.

Problem 24. Let $\mathcal{A}$ be a unital C^*-algebra, $\{\alpha_t\}_{t\in\mathbb{R}}$ be a one-parameter strongly continuous group of $*$-automorphisms of $\mathcal{A}$ (i.e., for any $A \in \mathcal{A}$, $t \to 0 \Rightarrow \|\alpha_t(A) - A\| \to 0$), $\beta \in \mathbb{R}\setminus\{0\}$, and ω be an (α_t, β)-KMS.

(i) Prove that ω is α_t-invariant, i.e., $\omega(\alpha_t(A)) = \omega(A)\ \forall A \in \mathcal{A}\ \forall t \in \mathbb{R}$.
(*Hint:* consider the function $\mathbb{C} \ni z \mapsto F(z) := \omega(\alpha_z(A))$ on each strip $\mathfrak{Im}\, z \in [n, n+1]$, $n \in \mathbb{Z}$.)

(ii) Give a concrete example of $\mathcal{A}$, α_t, and ω for which (i) ceases to be true if $\beta = 0$.

Problem 25. Let $\mathcal{A}$ be a unital C^*-algebra and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a one-parameter strongly continuous group of $*$-automorphisms of $\mathcal{A}$. Let ω be a state on $\mathcal{A}$. Prove that the following conditions are equivalent:

(1) ω satisfies the (α_t, β)-KMS condition,

(2) the relation
$$\int_{-\infty}^{+\infty} \mathrm{d}t\, f(t)\, \omega(A\alpha_t(B)) \;=\; \int_{-\infty}^{+\infty} \mathrm{d}t\, f(t+\mathrm{i}\beta)\, \omega(\alpha_t(B)A)$$
is valid for all $A, B \in \mathcal{A}$ and all f with Fourier transform $\widehat{f} \in C_0^\infty(\mathbb{R})$.

Problem 26. Let $\mathcal{A}$ be a unital C^*-algebra. Let $\{\alpha_t^{(n)}\}_{t\in\mathbb{R}}$, $n \in \mathbb{N}$, and $\{\alpha_t^{(n)}\}_{t\in\mathbb{R}}$ be one-parameter strongly continuous groups of $*$-automorphisms of $\mathcal{A}$ such that $\|\alpha_t^{(n)}(A) - \alpha_t(A)\| \to 0$ as $n \to \infty$ for all $A \in \mathcal{A}$ and $t \in \mathbb{R}$. Assume that $\{\omega_n\}_{n=1}^\infty$ is a sequence of $(\alpha_t^{(n)}, \beta)$-KMS states which converges in the weak-$*$ topology to a state ω. Prove that ω is a (α_t, β)-KMS state.

Problem 27. Consider the C^*-algebra $\mathcal{A} = \mathcal{M}_n(\mathbb{C})$ and a generic state on it, which necessarily has the form $\mathcal{A} \ni A \mapsto \omega_\rho(A) = \mathrm{Tr}(\rho A)$ for some density matrix ρ. Let $H = H^* \in \mathcal{A}$ and $\beta \in \mathbb{R}$, and correspondingly consider the Gibbs state ω_G given by the density matrix $\rho_G := \dfrac{e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}}$. Define an entropy function $\omega_\rho \mapsto S(\omega_\rho) := -\mathrm{Tr}\, \rho \log \rho \in [0, \log n]$. (The function $-x\log x$ is defined by continuity to be zero at $x = 0$.) Prove that ω_G is the unique state which maximizes the function
$$F(\omega_\rho) \;:=\; \frac{1}{\beta} S(\omega_\rho) - \omega_\rho(H)\,.$$
(*Hint:* use the convexity inequality $-\mathrm{Tr}(A\log A - A\log B) \leqslant \mathrm{Tr}(A - B)$ for Hermitian matrices A and B, and the identification $F(\omega_\rho) = -\beta^{-1}\mathrm{Tr}(\rho\log\rho - \rho\log\rho_G) + \beta^{-1}\log\mathrm{Tr}(e^{-\beta H})$.)

Problem 28. Let $\mathcal{A}$ be a unital C^*-algebra and let $\{\alpha_t\}_{t\in\mathbb{R}}$ be a one-parameter strongly continuous group of $*$-automorphisms of $\mathcal{A}$. Let δ be the infinitesimal generator of α_t (Problem 21), and assume that ω is a state on $\mathcal{A}$ such that $\omega(A\delta(A)) \in \mathrm{i}\mathbb{R}$ for all $A = A^* \in D_\delta$. Prove that ω is α_t-invariant, i.e., $\omega(\alpha_t(A)) = \omega(A)$ for all $A \in \mathcal{A}$ and $t \in \mathbb{R}$.