ADDITIONAL PROBLEMS, WEEK 01

Note: Additional problems are handed out every week as a supplement to the material discussed in class and in the homework. They are not part of the homework load, no solution has to be submitted, they bring no credit for the final mark. On the other hand, they are meant to provide further examples and applications to the notions presented in class, as well as further material for your own preparation at home for the final test.

Info: `www.math.lmu.de/~michel/SS13_MSP.html`

Problem 1. Let X be a locally compact space. Recall that $C_0(X)$ denotes the continuous functions over X, say f, such that for each f and $\varepsilon > 0$ there is a compact $K \subset X$ with $|f(x)| < \varepsilon$ for all $x \in X \setminus K$. Let $\mathcal{A} = C_0(X)$ be the commutative C^*-algebra obtained with the usual pointwise sum, product, and complex conjugation, and with the supremum norm. Let $F \subset X$ be a closed subset of X and let $\mathcal{I}$ be the collection of the elements in $\mathcal{A}$ which are zero on F.

(i) Prove that $\mathcal{I}$ is a closed ideal of $\mathcal{A}$.

(ii) Prove that the quotient algebra $\mathcal{A}/\mathcal{I}$ is identifiable as $C_0(F)$.

(iii) Prove that any closed, two-sided ideal in $\mathcal{A}$ has the form of some $C_0(F)$.

Problem 2. Let $\mathcal{A}$ be a unital Banach algebra. Show that if an element $A \in \mathcal{A}$ belongs to the open unit ball centred at $\mathbb{1}$, i.e., $\|A - \mathbb{1}\| < 1$, then A is invertible and A^{-1} is written as the norm-convergent series

$$A^{-1} = \sum_{n=0}^{\infty} (\mathbb{1} - A)^n ,$$

where $B^0 := \mathbb{1}$ for all $B \in \mathcal{A}$.

Problem 3. Let $\mathcal{A}$ be a unital Banach algebra and denote by $G(\mathcal{A})$ the group of all invertible elements in $\mathcal{A}$.

(i) Prove that the group $G(\mathcal{A})$ is an open subset of $\mathcal{A}$. More precisely, prove that if $\|A - A_0\| < 1/\|A_0^{-1}\|$ for an $A_0 \in G(A)$ then A is invertible and

$$A^{-1} = \Big(\sum_{n=0}^{\infty} \big(A_0^{-1}(A_0 - A) \big)^n \Big) A_0^{-1} .$$

(ii) Prove that the map $A \mapsto A^{-1}$ is a continuous map on $G(\mathcal{A})$.

Problem 4. Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$. As usual, denote by $\sigma(A)$ the spectrum of an element $A \in \mathcal{A}$, and by $r(A) := \sup_{\lambda\in\sigma(A)} |\lambda|$ the spectral radius of A.

(i) Prove that $\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}$.

(ii) Prove that $r(AB) = r(BA)$.

Problem 5. Let $\mathcal{A}$ be a unital Banach algebra. As usual, denote by $\sigma(A)$ the spectrum of an element $A \in \mathcal{A}$, and by $r(A) := \sup_{\lambda\in\sigma(A)} |\lambda|$ the spectral radius of A.

(i) Prove that $\sigma(A)$ is contained in the disk of radius $\|A\|$ in $\mathbb{C}$, i.e., prove that $\lambda \in \sigma(A) \Rightarrow |\lambda| \leqslant \|A\|$. (This also says that $r(A)$ is finite, $r(A) \leqslant \|A\|$.)

(ii) Prove that $\sigma(A)$ is a closed subset of $\mathbb{C}$. (This, together with (i), also says that $\sigma(A)$ is compact in $\mathbb{C}$.)

Problem 6. Let $\mathcal{A}$ be a unital Banach algebra. As usual, denote by $\sigma(A)$ the spectrum of an element $A \in \mathcal{A}$, and by $\rho(A)$ the resolvent set of A.

(i) Prove that the map $\rho(A) \ni \lambda \mapsto (\lambda - A)^{-1} \in \mathcal{A}$ is continuous.

(ii) Prove that the map $\rho(A) \ni \lambda \mapsto (\lambda - A)^{-1} \in \mathcal{A}$ is holomorphic.

(iii) Prove that $\sigma(A)$ is non-empty. (*Hint:* assume for contradiction that $\rho(A) = \mathbb{C}$ and prove and use that $\|(\lambda - A)^{-1}\| \to 0$ as $|\lambda| \to \infty$.)

Note that it is essential for the proof of (iii) that the field of scalars of our Banach algebra is the complex field.

Problem 7. Let $\mathcal{A}$ be a unital Banach algebra. As usual, denote by $r(A) := \sup_{\lambda\in\sigma(A)} |\lambda|$ the spectral radius of an element $A \in \mathcal{A}$. Prove that

$$r(A) \;=\; \lim_{n\to\infty} \|A^n\|^{1/n} \;=\; \inf_n \|A^n\|^{1/n}\,.$$

Problem 8. Let $\mathcal{A}$ be a unital Banach algebra. Let $A \in \mathcal{A}$. As usual, denote by $\sigma(A)$ the spectrum of $A \in \mathcal{A}$.

(i) Let p be a polynomial in a single variable. Prove that

$$\sigma(p(A)) \;=\; p(\sigma(A)) \;:=\; \{p(\lambda)\,|\,\lambda \in \sigma(A)\}\,.$$

(ii) Assume that A is invertible. Prove that

$$\sigma(A^{-1}) \;=\; \sigma(A)^{-1} \;:=\; \{\lambda^{-1}\,|\,\lambda \in \sigma(A)\}\,.$$

(iii) Assume that $\mathcal{A}$ be a unital B^*-algebra. Prove that

$$\sigma(A^*) \;=\; \overline{\sigma(A)} \;:=\; \{\overline{\lambda}\,|\,\lambda \in \sigma(A)\}\,.$$

Problem 9. Let $\mathcal{A}$ be a unital C^*-algebra. As usual, denote by $r(A)$ the spectral radius of an element $A \in \mathcal{A}$.

(i) Assume that $AA^* = A^*A$. Prove that $r(A) = \|A\|$.

(ii) Assume that $AA^* = A^*A = \mathbb{1}$. Prove that $\sigma(A) \subset \{\lambda \in \mathbb{C} \,|\, |\lambda| = 1\}$.

(iii) Assume that $A = A^*$. Prove that $\sigma(A) \subset [-\|A\|, \|A\|]$ and that $\sigma(A^2) \subset [0, \|A\|^2]$.

Problem 10. Let $\mathcal{A}$ be a unital algebra and let $\mathcal{B} \subset \mathcal{A}$ be a unital subalgebra of $\mathcal{A}$. Let $B \in \mathcal{B}$. Note that there are two possible spectra of B, $\sigma_{\mathcal{A}}(B)$ and $\sigma_{\mathcal{B}}(B)$, depending on whether the invertibility of $(B - \lambda\mathbb{1})$ is referred to $\mathcal{A}$ or $\mathcal{B}$.

(i) Prove that $\sigma_{\mathcal{A}}(B) \subset \sigma_{\mathcal{B}}(B)$.

(ii) Assume in addition that $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{B} \subset \mathcal{A}$ is a unital C^*-subalgebra of $\mathcal{A}$. Prove that $\sigma_{\mathcal{A}}(B) = \sigma_{\mathcal{B}}(B)$.

(iii) Produce an explicit example of $\mathcal{A}$ and $\mathcal{B}$ as above with $\sigma_{\mathcal{A}}(B) \subsetneq \sigma_{\mathcal{B}}(B)$.

Problem 11. Let $\mathcal{A}$ be a Banach algebra.

(i) Assume that $\mathcal{I} \subset \mathcal{A}$ is a proper (left or right) ideal in $\mathcal{A}$. Prove that the norm closure $\overline{\mathcal{I}}$ of $\mathcal{I}$ is a proper (left or right) ideal in $\mathcal{A}$.

(ii) Assume that $\mathcal{I}$ is a maximal (left, right, or two-sided) ideal in $\mathcal{A}$. Prove that $\mathcal{I}$ is norm-closed.

Problem 12. Let $\mathcal{A}$ be a unital commutative Banach algebra. For each $A \in \mathcal{A}$ denote as usual by $\sigma(A)$ and $r(A)$ respectively the spectrum and the spectral radius of A. Let $A, B \in \mathcal{A}$. Prove that

$$\begin{aligned} \sigma(AB) &\subset \sigma(A)\sigma(B)\,, & \sigma(A+B) &\subset \sigma(A)+\sigma(B)\,, \\ r(AB) &\leqslant r(A)r(B)\,, & r(A+B) &\leqslant r(A)+r(B)\,. \end{aligned}$$

Problem 13.

(i) Prove that any commutative Banach algebra possesses at least one non-zero multiplicative (bounded linear) functional.

(ii) Produce an explicit example where without the assumption of commutativity the result (i) fails to hold, i.e., a non-commutative Banach algebra that has no non-zero multiplicative (bounded linear) functional.

Problem 14. Prove that the Banach space $L^1(\mathbb{R})$ equipped with the convolution product $f * g$ defined by

$$(f * g)(x) \;:=\; \int_{-\infty}^{+\infty} f(x-y)g(y)\,\mathrm{d}y \qquad \text{for a.e. } x \in \mathbb{R} \qquad (f, g \in L^1(\mathbb{R}))$$

is a non-unital commutative Banach algebra. Can the same be said for $L^2(\mathbb{R})$?

Problem 15. Let $\mathcal{A}$ be a unital C^*-algebra and let $A \in \mathcal{A}$. Denote by $\mathcal{A}_A$ the closure in $\mathcal{A}$ of the $*$-algebra of complex polynomials in A, A^*, and $\mathbb{1}$ (the so-called "C^*-ALGEBRA GENERATED BY A).

(i) Prove that $\mathcal{A}_A$ is the smallest (in the sense of inclusion) C^*-subalgebra of $\mathcal{A}$ containing A, A^*, and $\mathbb{1}$.

(ii) Prove that $\mathcal{A}_A$ is commutative if and only if $AA^* = A^*A$.

(iii) Assume in addition that $A \in \mathcal{A}$ is invertible. Prove that $A^{-1} \in \mathcal{A}_A$.

Problem 16. Prove that the norm on a C^*-algebra is unique.

Problem 17. Let $\mathcal{A}$ be a commutative unital Banach algebra. The space

$$\mathrm{Rad}(\mathcal{A}) \;:=\; \bigcap_{\substack{\mathcal{I} \text{ maximal} \\ \text{ideal in } \mathcal{A}}} \mathcal{I}$$

is an ideal of $\mathcal{A}$ itself and is called the RADICAL OF $\mathcal{A}$.

(i) Prove that if $A \in \mathcal{A}$ is nilpotent, then $A \in \mathrm{Rad}(\mathcal{A})$.

(ii) Calculate the radical of a commutative unital C^*-algebra. Prove that a commutative unital C^*-algebra contains no nilpotent elements.

Problem 18. Let X be a compact topological space. Define the map

$$\pi : X \to M(C(X))\,, \qquad x \mapsto \delta_x\,,$$

where $\delta_x(f) := f(x)$ for all $f \in C(X)$. Prove that π is a homeomorphism, i.e., a continuous bijective map with continuous inverse.

(*Hint:* to prove surjectivity show that if there is $m \in M(C(X)) \setminus \pi(X)$ then using the compactness of X one can construct $f \in C(X)$ with $f > 0$ and $m(f) = 0$.)