HOMEWORK ASSIGNMENT 11

Hand-in deadline: Tuesday 9 July 2013 by 4 p.m. in the "MSP" drop box.

Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.

Info: `www.math.lmu.de/~michel/SS13_MSP.html`

Exercise 41. (Coherent states in the bosonic Fock space)

Given a Hilbert space $\mathfrak{h}$, consider the bosonic Fock space $\mathfrak{F}_+(\mathfrak{h})$ with the usual notation for the annihilation, creation, and number operator, respectively $a(f)$, $a^*(f)$, $\mathcal{N}$, where $f \in \mathfrak{h}$. Define $W(f) := e^{\overline{a^*(f)-a(f)}}$. (Note that, apart from an irrelevant normalisation, $W(f)$ is nothing but the Weyl operator defined in class associated with the function $\mathrm{i}f$.) Denote by Ω the vacuum in the Fock space.

(i) Given $f \in \mathfrak{h}$, the state $W(f)\Omega \in \mathfrak{F}_+(\mathfrak{h})$ is usually called COHERENT STATE with one-particle state f. Prove that

$$W(f)\Omega \;=\; e^{-\|f\|^2/2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}\,f^{\otimes n}\,,$$

where $f^{\otimes n}$ indicates the Fock-vector $\{0,\dots,f^{\otimes n},0,\dots\}$ and $\|\;\|$ is the norm in $\mathfrak{h}$.

Remark: this justifies the terminology of "coherent state": $W(f)\Omega$ is a superposition of states with a different number of particles, the probability of having n particles in $W(f)\Omega$ being given by $e^{-\|f\|^2}/n!$.

(ii) Prove that the expectation of the number of particles in the coherent state of f is $\|f\|^2$, namely prove that

$$\big\langle W(f)\Omega\,,\,\mathcal{N}\,W(f)\Omega\big\rangle_{\mathfrak{F}(\mathfrak{h})} \;=\; \|f\|^2 \;=\; \sum_{n=1}^{\infty}\big\langle W(f)\Omega, a^*(f_n)a(f_n)W(f)\Omega\big\rangle_{\mathfrak{F}(\mathfrak{h})}\,,$$

where the second identity is understood under the additional assumption that $\{f_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathfrak{h}$.

(iii) Let $N \in \mathbb{N}$ and $f \in \mathfrak{h}$ with $\|f\| = 1$. Consider the factorised N-particle state

$$\Psi_N \;:=\; \{0,\dots,0,f^{\otimes N},0,0,\dots\} \;\in\; \mathfrak{F}_+(\mathfrak{h})\,.$$

Prove that Ψ_N can be expressed as the following linear superposition of coherent states:

$$\Psi_N \;=\; C_N\int_0^{2\pi}\frac{\mathrm{d}\theta}{2\pi}\,e^{i\theta N}W(e^{-\mathrm{i}\theta}\sqrt{N}f)\Omega$$

with the constant $C_N := \dfrac{\sqrt{N!}}{N^{N/2}e^{-N/2}}$. (Note that $C_N \sim N^{1/4}$ as $N \to \infty$.)

Remark: the above expansion is useful in the study of the time evolution of a *Bose-Einstein condensate* of N identical bosons all prepared at time zero in the state f, because it reduces the study of the evolution of Ψ_N to the evolution of each coherent state, which is somewhat more manageable in the Fock space formalism. (Why? Because the dynamics tend not to keep the particle number fixed!) See, for instance,

– I. Rodnianski, B. Schlein. *Quantum fluctuations and rate of convergence towards mean field dynamics.* Comm. Math. Phys. 291 (2009), 31–61

– A. Michelangeli, B. Schlein. *Dynamical collapse of boson stars*, Comm. Math. Phys. 311 (2012), 645–687

Exercise 42. (Group of Bogoliubov transformations for the ideal Bose gas.)

Given a Hilbert space $\mathfrak{h}$, consider the bosonic Fock space $\mathfrak{F}_+(\mathfrak{h})$ with the usual notation for the second quantisation, annihilation and creation operators, respectively Γ, $a(f)$, $a^*(f)$, where $f \in \mathfrak{h}$. Define $W(f) := e^{\overline{a^*(f)-a(f)}}$. (Note that, apart from an irrelevant normalisation, $W(f)$ is nothing but the Weyl operator defined in class associated with the function $\mathrm{i}f$.) Let H be a one-particle Hamiltonian on $\mathfrak{h}$. Define

$$\mathcal{L}(\mathfrak{F}_+(\mathfrak{h})) \ni A \mapsto \tau_t(A) := \Gamma(e^{\mathrm{i}tH})A\,\Gamma(e^{-\mathrm{i}tH}) \in \mathcal{L}(\mathfrak{F}_+(\mathfrak{h}))\,, \qquad t \in \mathbb{R}\,.$$

(i) Prove that
$$\tau_t(W(f)) = W(e^{\mathrm{i}tH}f)$$
for each $t \in \mathbb{R}$ and each $f \in \mathfrak{h}$. (Whence τ_t leaves $\mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})$ invariant.)
(*Hint:* on the L.H.S. re-write $W(f)$ expanding $e^{a^*(f)-a(f)}$ by means of $[a(f), a^*(f)] = \|f\|^2$; on the R.H.S. compute $a(e^{\mathrm{i}tH}f)$ and $a^*(e^{\mathrm{i}tH}f)$ in terms of $a(f)$ and $a^*(f)$ respectively, analogously to what was done in class in the fermionic case.)

(ii) Prove that $\{\tau_t\}_{t\in\mathbb{R}}$ is a one-parameter group of $*$-automorphism of the CCR algebra in $\mathcal{L}(\mathfrak{F}_+(\mathfrak{h}))$ generated by W.

(iii) Is the group $\{\tau_t\}_{t\in\mathbb{R}}$ strongly continuous? Justify your answer.

Exercise 43. (Distributional formalism for the (bosonic) Fock space)

Given the Hilbert space $\mathfrak{h} = L^2(\mathbb{R}^d)$, consider the bosonic Fock space $\mathfrak{F}_+(\mathfrak{h})$ with the usual notation for the annihilation, creation, and number operator, respectively $a(f)$, $a^*(f)$, $\mathcal{N}$, where $f \in \mathfrak{h}$. Introduce the *operator-valued distributions* a_x^* and a_x, with $x \in \mathbb{R}^d$, defined so that

$$\begin{aligned} a^*(f) &= \int_{\mathbb{R}^d} \mathrm{d}x\, f(x)\, a_x^*\,, \\ a(f) &= \int_{\mathbb{R}^d} \mathrm{d}x\, \overline{f(x)}\, a_x \end{aligned}$$

for every $f \in L^2(\mathbb{R}^d)$, namely whose explicit action is

$$\begin{aligned} (a_x\Psi)^{(n)}(x_1,\dots,x_n) &:= \sqrt{n+1}\,\Psi^{(n+1)}(x,x_1,\dots,x_n)\,, \\ (a_x^*\Psi)^{(n)}(x_1,\dots,x_n) &:= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\delta(x-x_i)\Psi^{(n-1)}(x_1,\dots,\widehat{x}_i\dots,x_n)\,, \end{aligned}$$

for every $\Psi \in \mathfrak{F}_+(\mathfrak{h})$. (The notation $\widehat{x}_i$ means that the variable x_i is missing.)

(i) Prove that the CCR for a_x^* and a_x take the form

$$[a_x, a_y^*] \;=\; \delta(x-y)\,, \qquad [a_x, a_y] \;=\; [a_x^*, a_y^*] \;=\; 0\,.$$

(ii) Prove that the number operator, expressed through the distributions a_x^* and a_x, is given by

$$\mathcal{N} \;=\; \int_{\mathbb{R}^d} \mathrm{d}x\, a_x^* a_x\,.$$

(iii) Let $d \in \mathbb{N}$. Consider a smooth function $V : \mathbb{R}^d \to \mathbb{R}$. Prove that

$$\mathcal{H} \;:=\; \int_{\mathbb{R}^d} \mathrm{d}x\, a_x^*(-\Delta_x) a_x \;+\; \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{d}x\,\mathrm{d}y\, V(x-y)\, a_x^* a_y^* a_y a_x$$

defines an operator acting on $\mathfrak{F}_+(\mathfrak{h})$ that on the n-particle sector $L^2(\mathbb{R}^{nd})$, $n \in \mathbb{N}$, acts as

$$\mathcal{H}^{(n)} \;=\; \sum_{j=1}^{n}(-\Delta_{x_j}) \;+ \sum_{1\leqslant i<j\leqslant n} V(x_i - x_j) \qquad (x_j \in \mathbb{R}^d)\,.$$

Exercise 44. (Two-point function for a bosonic Gibbs state)

Given a Hilbert space $\mathfrak{h}$, consider the CCR algebra $\mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})$ acting on the bosonic Fock space $\mathfrak{F}_+(\mathfrak{h})$ and the Gibbs state ω acting on $\mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})$ as

$$\omega(A) \;:=\; \frac{\mathrm{Tr}(e^{-\beta\,\mathrm{d}\Gamma(H-\mu\mathbb{1})}A)}{\mathrm{Tr}(e^{-\beta\,\mathrm{d}\Gamma(H-\mu\mathbb{1})})}\,, \qquad \beta,\mu\in\mathbb{R}\,, \quad A \in \mathcal{A}_{\mathrm{CCR}}(\mathfrak{h})\,,$$

where Γ is the second quantisation operator and H is one-particle Hamiltonian on $\mathfrak{h}$ *such that* $\beta(H-\mu\mathbb{1}) > \mathbb{O}$. Consider the two-point functions $\omega(a^*(f)a(g))$ with $f,g\in\mathfrak{h}$.

(i) Prove that $\omega(a^*(f)a(g))$ is well defined and there is a constant $C_{\beta,\mu}$ such that

$$|\omega(a^*(f)a(g))| \;\leqslant\; C_{\beta,\mu}\,\|f\|\,\|g\| \qquad \forall f,g\in\mathfrak{h}\,.$$

(ii) Let $n\in\mathbb{N}$ and $f_1,\dots,f_n,g_1,\dots,g_n\in\mathfrak{h}$. Mimicking the analogous calculation done in class for the fermionic case, but of course applying now the *commutation* relations, compute

$$\omega\Big(\prod_{i=1}^{n} a^*(f_i)\prod_{j=1}^{n} a(g_j)\Big)$$

and deduce that the value of ω on monomials of the a and a^* are determined by sums of products of the above two-point function.

Hints

Recommendation: try first to solve the exercises with the only amount of information provided in their formulation. I.e., try to understand the question, to identify what the involved notions from class are, to structure a potentially successful solving strategy. Go through these additional hints only if you get completely stuck in your first attempts.

Hints for Exercise 41. (i) Use $[a(f), a^*(f)] = \|f\|^2$, which commutes with $a(f)$ and $a^*(f)$, to split the exponential $e^{a*(f)-a(f)}$. (ii) A standard Fock-space computation. (iii) Plug into the integral the expansion found in (i).

Hints for Exercise 42. (i) On the L.H.S. re-write $W(f)$ expanding $e^{a^*(f)-a(f)}$ by means of $[a(f), a^*(f)] = \|f\|^2$, which commutes with $a(f)$ and $a^*(f)$; on the R.H.S. prove and use that $a(e^{\mathrm{i}tH}f) = e^{\mathrm{i}t\mathrm{d}\Gamma(H)}a(f)e^{-\mathrm{i}t\mathrm{d}\Gamma(H)}$ and $a^*(e^{\mathrm{i}tH}f) = e^{\mathrm{i}t\mathrm{d}\Gamma(H)}a^*(f)e^{-\mathrm{i}t\mathrm{d}\Gamma(H)}$. One has to deal with the exponential of an *unbounded* self-adjoint operator: while it is false that the corresponding formal series converges, it is true that it does on a dense of *analytic vectors*. (ii) Direct check. (iii) It suffices (why?) to prove that $\|\tau_t(W(f)) - W(f)\| \geqslant C > 0$ for all $t \in \mathbb{R}$. To this aim, compute the L.H.S. using the result from (i) for τ_t, the multiplication properties of the Weyl operator $W(f)$, and the fact that $\|W(f) - \mathbb{1}\| = 2$ for all non-zero $f \in \mathfrak{h}$, whence also (why?) $\|e^{\mathrm{i}\alpha}W(f) - \mathbb{1}\| = 2$ for all $\alpha \in \mathbb{R}$ and all non-zero $f \in \mathfrak{h}$.

Hints for Exercise 43. The *formal* check is a straightforward direct check. It is convenient to perform it on each level of the Fock space at fixed number of particles.

Hints for Exercise 44. (i) A Cauchy-Schwarz inequality together with the fact that $\mathrm{Tr}(A^*A) \leqslant C_{\beta,\mu}\|f\|^2$ where $A := a(f)e^{-\beta\,\mathrm{d}\Gamma(H-\mu\mathbb{1})/2}$. (ii) Prove and use that $a^*(e^{\mathrm{i}tH}f) = e^{\mathrm{i}t\mathrm{d}\Gamma(H)}a^*(f)e^{-\mathrm{i}t\mathrm{d}\Gamma(H)}$. This, together with the CCRs and the relation $\omega(a^*(f)a(g)) = \langle g\,, e^{-\beta H}(\mathbb{1}+e^{-\beta H})^{-1}f\rangle$, should yield

$$\omega\Big(\prod_{i=1}^{n} a^*(f_i)\prod_{j=1}^{n} a(g_j)\Big) \;=\; \sum_{p=1}^{n}\omega\big(a^*(f_1)a(g_p)\big)\,\omega\Big(\prod_{i=2}^{n} a^*(f_i)\prod_{\substack{j=1\\ j\neq p}}^{n} a(g_j)\Big)\,.$$

Iterate it to get the conclusion.