Mathematical Statistical Physics

TMP Programme Munich – spring term 2013

HOMEWORK ASSIGNMENT 08
Hand-in deadline: Tuesday 18 June 2013 by 4 p.m. in the "MSP" drop box.
Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/SS13_MSP.html`

Exercise 29. (Explicit matrix computations)
Let $\epsilon, \beta, \theta \in \mathbb{R}$. Let $J, K, A, B \in \mathcal{M}_2(\mathbb{C})$ be defined by

$$J := \begin{pmatrix} \cosh\theta & i\sinh\theta \\ -i\sinh\theta & \cosh\theta \end{pmatrix} \quad \text{and} \quad A := \cosh(2\beta\epsilon) \begin{pmatrix} \cosh(2\theta) & i\sinh(2\theta) \\ -i\sinh(2\theta) & \cosh(2\theta) \end{pmatrix},$$

$$K := \begin{pmatrix} \cosh(2\beta\epsilon) & i\sinh(2\beta\epsilon) \\ -i\sinh(2\beta\epsilon) & \cosh(2\beta\epsilon) \end{pmatrix} \quad \text{and} \quad B := \sinh(2\beta\epsilon) \begin{pmatrix} -\frac{1}{2}\sinh(2\theta) & -i\sinh^2\theta \\ i\cosh^2\theta & -\frac{1}{2}\sinh(2\theta) \end{pmatrix}.$$

Further, define

$$\Delta := \begin{pmatrix} J & \mathbf{0} & \dots & \\ \mathbf{0} & J & & \\ \vdots & & \ddots & \\ \vdots & & & J \end{pmatrix} \in \mathcal{M}_{2^n}(\mathbb{C}), \qquad \text{where} \quad \mathbf{0} \in \mathcal{M}_2(\mathbb{C}),$$

$$\chi^{\pm} := \begin{pmatrix} \cosh(2\beta\epsilon) & 0 \ \dots \ 0 & \pm i\sinh(2\beta\epsilon) \\ 0 & & 0 \\ \vdots & \mathbf{K} & \vdots \\ 0 & & \\ \mp i\sinh(2\beta\epsilon) & 0 \ \dots \ 0 & \cosh(2\beta\epsilon) \end{pmatrix} \in \mathcal{M}_{2^n}(\mathbb{C}), \quad \text{where}$$

$$\mathbf{K} := \begin{pmatrix} K & \mathbf{0} & \dots & \\ \mathbf{0} & K & & \\ \vdots & & \ddots & \\ \vdots & & & K \end{pmatrix} \in \mathcal{M}_{2^{n-1}}(\mathbb{C}) \ .$$

(i) Prove that

$$\omega^{\pm} := \Delta\chi^{\pm}\Delta = \begin{pmatrix} A & B & 0 & 0 & \dots & 0 & \mp B^* \\ B^* & A & B & 0 & & 0 & 0 \\ 0 & B^* & A & B & & & \vdots \\ \vdots & & & & & & \vdots \\ 0 & 0 & & & & A & B \\ \mp B & 0 & & & & B^* & A \end{pmatrix} \in \mathcal{M}_{2^n}(\mathbb{C}) \ .$$

Denote the n complex roots of the equation $z^n + 1 = 0$ by $z_k := e^{\frac{i\pi k}{n}}$, $k = 1, 3, \cdots, 2n-1$.

(ii) Prove that $\det\left(A + z_k B + z_k^{-1} B^*\right) = 1$ for $k = 1, 3, \cdots, 2n-1$.

(iii) Prove that $0 \leqslant \mathrm{Tr}\left(A + z_k B + z_k^{-1} B^*\right) \leqslant C$ for $k = 1, 3, \cdots, 2n-1$, where the constant C depends only on β, ϵ, and θ.

Exercise 30. (Computation of the free energy density)

In class we saw that

$$\lim_{N\to\infty} \frac{1}{N} \log Q_I(0,T) \;=\; \frac{1}{2}\log\left[2\sinh(2\epsilon\beta)\right] + \mathcal{L}\,,$$

where

$$\mathcal{L} \;:=\; \frac{1}{2\pi^2}\int_0^\pi\int_0^\pi \mathrm{d}v\mathrm{d}v' \log\Big(2\cosh(2\beta\epsilon)\coth(2\beta\epsilon) - 2(\cos v + \cos v')\Big)\,.$$

(i) Prove that

$$\mathcal{L} \;=\; \frac{1}{2\pi^2}\iint_{\mathcal{R}} \mathrm{d}v\mathrm{d}v' \log\Big(2\cosh(2\beta\epsilon)\coth(2\beta\epsilon) - 2(\cos v + \cos v')\Big)$$

where the new region of integration $\mathcal{R}$ is the dashed rectangle shown in Figure 1.

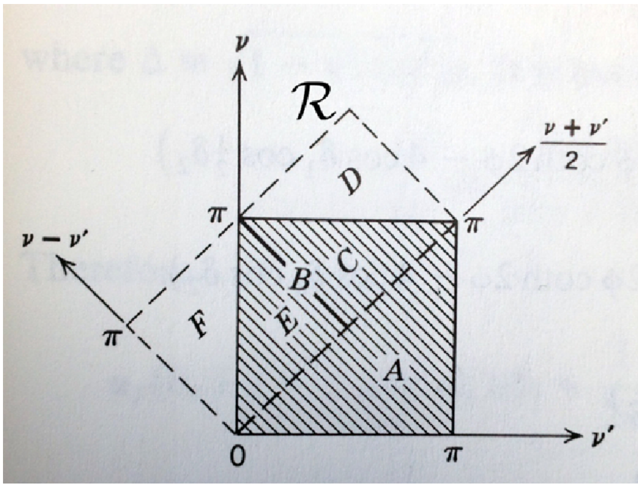

Figure 1: Change the region of integration.

(ii) Prove that

$$\mathcal{L} \;=\; \frac{1}{2\pi^2}\int_0^\pi\int_0^\pi \mathrm{d}w\ \mathrm{d}w'\ \log\left[2D - 4\cos(w)\cos\left(\frac{w'}{2}\right)\right]$$

where $D := \cosh(2\beta\epsilon)\coth(2\beta\epsilon)$.

(*Hint:* change of variables $\mathcal{R} \to [0,\pi]\times[0,\pi]$, $(v,v') \mapsto (\frac{v+v'}{2}, v - v') =: (w, w')$.)

(iii) Prove that

$$|z| \;=\; \frac{1}{\pi}\int_0^\pi \mathrm{d}t\ \log\left(2\cosh z - 2\cos t\right), \qquad z \in \mathbb{R}.$$

(*Hint:* Make use of $\int_0^\pi \log(a \pm b\cos x)\,\mathrm{d}x = \pi\log\left(\frac{a+\sqrt{a^2-b^2}}{2}\right)$, $a \leqslant b$, and $\mathrm{arccosh}(x) = \log(x + \sqrt{x^2-1})$, $x \geqslant 1$.)

(iv) Prove that

$$\mathcal{L} \;=\; \frac{1}{2}\log(2D) + \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}s \;\log\frac{1}{2}\left(1+\sqrt{1-\kappa^2\sin^2 s}\right)\,, \qquad \kappa := \frac{2}{D}\,.$$

(*Hint:* show first that

$$\mathcal{L} \;=\; \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}w' \log\left[2\cos\left(\frac{w'}{2}\right)\right] + \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}w' \;\operatorname{arccosh}\left(\frac{D}{2\cos\frac{w'}{2}}\right)$$

and use $\operatorname{arccosh}(x) = \log(x+\sqrt{x^2-1})$, $x \in \mathbb{R}$.)

In 1936, eight years before Onsager's exact solution, Peierls gave a (non-rigorous) proof that the two-dimensional Ising model exhibits a phase transition. His key idea was to consider the boundary line between regions of positive and negative spin and to relate it to the magnetisation. The following exercise uses a version of Peierls' argument to study the magnetisation in the finite volume case with positive boundary conditions. In next week's assignment this will be used to prove the emergence of the non-analyticity of the free energy in the thermodynamic limit, hence the existence of a phase transition.

Exercise 31. (Peierls' argument)

Consider the Ising model on the square lattice $\Lambda \subseteq \mathbb{R}^2$ with n rows and n columns; let $N = n^2$ be the number of lattice points. We impose positive boundary conditions, i.e., every spin outside Λ is supposed to have value $+1$. This is reflected in the modified energy function

$$E_+(S) \;=\; -\epsilon \sum_{\langle pq\rangle : p,q\in\Lambda} s_p s_q - \epsilon \sum_{\langle pq\rangle : p\in\Lambda, q\in\Lambda^c} s_p - B\sum_{p\in\Lambda} s_p,$$

where $S = (s_p)_{p\in\Lambda}$ denotes any configuration of spins in Λ and the notation $\langle pq\rangle$ means pairs of directly neighbouring sites p and q. Let $B = 0$. The corresponding magnetisation per site at inverse temperature β is given by

$$m_{\Lambda,+}(\beta, B=0) \;=\; \langle \frac{1}{N}\sum_{p\in\Lambda} s_p\rangle_+ = \frac{1}{N}\sum_{p\in\Lambda}\langle s_p\rangle_+$$

with

$$\langle s_p\rangle_+ \;=\; \frac{\sum_{S:s_p=+1} \mathrm{e}^{-\beta E_+(S)}}{\sum_S \mathrm{e}^{-\beta E_+(S)}} - \frac{\sum_{S:s_p=-1} \mathrm{e}^{-\beta E_+(S)}}{\sum_S \mathrm{e}^{-\beta E_+(S)}}.$$

Given a configuration S, we can look at the perpendicular border between regions of positive and negative spin (see Fig. 2 for a self-explanatory definition). Owing to the positive boundary conditions, the border is given as a union of closed circles which we will call *contours*. The length $|\gamma|$ of a contour γ is defined to be the number of elementary edges it consists of. For $p \in \Lambda$ and γ a contour, we write $\gamma \circlearrowleft p$ if p is surrounded by γ. If γ appears in a configuration S, we write $\gamma \in S$.

(i) Prove that

$$\langle s_p\rangle_+ \;\geqslant\; 1 - 2\frac{\sum_{S:\exists\gamma\circlearrowleft p} \mathrm{e}^{-\beta E_+(S)}}{\sum_S \mathrm{e}^{-\beta E_+(S)}} \qquad \forall p \in \Lambda$$

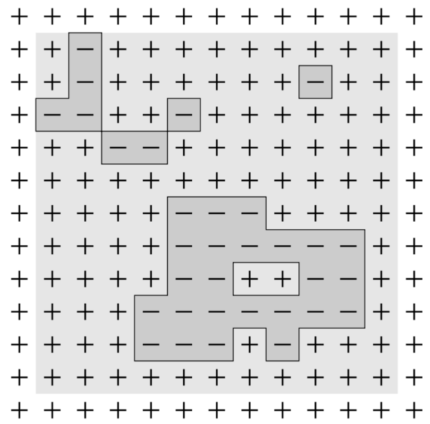

Figure 2: Spin configuration with corresponding contours

(ii) Prove that for any contour γ

$$\frac{\sum_{S:\gamma\in S} \mathrm{e}^{-\beta E_+(S)}}{\sum_S \mathrm{e}^{-\beta E_+(S)}} \leqslant \mathrm{e}^{-2\beta\epsilon|\gamma|}\,.$$

(*Hint:* note that $\displaystyle\sum_{S:\exists\gamma\circlearrowleft p} \mathrm{e}^{-\beta E_+(S)} \leqslant \sum_{\gamma:\gamma\circlearrowleft p}\sum_{S:\gamma\in S} \mathrm{e}^{-\beta E_+(S)}$.)

(iii) Prove that

$$\big(\text{number of contours of length } l \text{ that surround } p\big) \leqslant \frac{l}{2}\,3^{l-2}\,.$$

(iv) Conclude that if β is large enough then $m_{\Lambda,+}(\beta, B=0) \geqslant \frac{1}{2}\ \forall N$.

Exercise 32. (Free slot! Thus, only three homework exercises for this week.)

Hints

Recommendation: try first to solve the exercises with the only amount of information provided in their formulation. I.e., try to understand the question, to identify what the involved notions from class are, to structure a potentially successful solving strategy. Go through these additional hints only if you get completely stuck in your first attempts.

Hints for Exercise 29.
(i) Note that $\chi^{\pm}$ can be written in terms of block matrices

$$\chi^{\pm} = \begin{pmatrix} \tilde{A} & \tilde{B} & 0 & 0 & \dots & 0 & \mp\tilde{B}^* \\ \tilde{B}^* & \tilde{A} & \tilde{B} & 0 & & 0 & 0 \\ 0 & \tilde{B}^* & \tilde{A} & \tilde{B} & & & \vdots \\ \vdots & & & & & & \vdots \\ & & & & & & \\ 0 & 0 & & & & \tilde{A} & \tilde{B} \\ \mp\tilde{B} & 0 & & & & \tilde{B}^* & \tilde{A} \end{pmatrix} \in \mathcal{M}_{2^n}(\mathbb{C}),$$

where you should find suitable $\tilde{A}, \tilde{B} \in \mathcal{M}_2(\mathbb{C})$. (ii) Use $\sinh(2x) = 2\sinh x \cosh x, \cosh(2x) = 2\sinh^2 x + 1$, and $\cosh^2 x = 1 + \sinh^2 x$. Note that $z_k + z_k^{-1} = 2\cos(\pi k/n)$. (iii) Use $\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$ and note that $\cosh x \geq 0$ for all $x \in \mathbb{R}$.

Hints for Exercise 30.
(i) Show that $\int_A = \int_B$, $\int_C = \int_D$, and $\int_E = \int_F$.

Hints for Exercise 31.
A general note: all estimates in this exercise are relatively crude, so don't be reluctant to drop terms etc. (i) Note that if $s_p = -1$, there has to be at least one contour surrounding p. (ii) If S is a configuration such that $\gamma \in S$, compare it with the configuration $\tilde{S}$ obtained by flipping all spins inside γ. (iii) Easy combinatorics. (iv) Put everything together.