HOMEWORK ASSIGNMENT 05
Hand-in deadline: Tuesday 28 May 2013 by 4 p.m. in the "MSP" drop box.
Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/SS13_MSP.html`

In both Exercise 17 and Exercise 18 you will need the following Hahn-Banach theorem: *Let Y be a subspace of a normed linear space X and let ϕ be a bounded linear functional on Y. Then ϕ has a bounded linear extension Φ on the whole X such that $\|\Phi\|_{X^*} = \|\phi\|_{Y^*}$.*

Exercise 17.

(i) Let $\mathcal{A}$ be a unital C^* algebra, $\mathcal{B} \subset \mathcal{A}$ be a C^*-subalgebra with the same unit, and ω be a state on $\mathcal{B}$. Prove that ω has an extension to a state on $\mathcal{A}$, i.e., there exists a state ρ on $\mathcal{A}$ such that $\rho\big|_{\mathcal{B}} = \omega$.

(ii) Prove that the set of states on a unital C^*-algebra $\mathcal{A}$ separates points, i.e., for any $A, B \in \mathcal{A}$ with $A \neq B$ there exists a state ω on $\mathcal{A}$ with $\omega(A) \neq \omega(B)$.

Exercise 18. Let $\mathcal{A}$ be a unital C^*-algebra and let $A \in \mathcal{A}$.

(i) Prove that there exists a pure state ω on $\mathcal{A}$ such that $\omega(A^*A) = \|A\|^2$.
(*Hint:* On the elements of the form $\alpha\mathbb{1} + \beta A^*A$ consider the map $\phi(\alpha\mathbb{1} + \beta A^*A) := \alpha + \beta\|A\|^2$. Then apply the Krein-Milman theorem to the set of states on $\mathcal{A}$ with the property $\omega(A^*A) = \|A\|^2$.)

(ii) Prove that there exists an irreducible representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$ such that $\|\pi(A)\| = \|A\|$.

Exercise 19. Consider the following:

(i) a finite quantum spin system (namely a system of spins on a finite lattice, as discussed in tutorial) and the corresponding C^*-algebra $\mathcal{A}$ that contains the observables of the system;

(ii) an infinite quantum spin system and the corresponding quasi-local C^*-algebra $\mathcal{A}$ that contains the observables of the system.

In each case, give an explicit example of a time evolution, i.e., a one-parameter continuous group $\{\alpha_t\}_{t\in\mathbb{R}}$ of automorphisms on $\mathcal{A}$, such that $\mathcal{A}$ admits α_t-invariant states that are *not* α_t-KMS states.

(You are encouraged not to be completely trivial...)

Exercise 20. The purpose of this exercise is to invert the result contained in Proposition 2.4.10 given in class.

Let $\mathcal{A}$ be a unital C^*-algebra and let ω be a bounded linear functional on $\mathcal{A}$ such that $\|\omega\|_{\mathcal{A}^*} = \omega(\mathbb{1}) = 1$. Prove that ω is a state.

(*Hint:* Show first that $A = A^* \Rightarrow \omega(A) \in \mathbb{R}$ by considering elements of the form $A + \mathrm{i}\gamma\mathbb{1}$, $\gamma \in \mathbb{R}$. Then prove positivity of ω using Exercise 12.)

Hints

Recommendation: try first to solve the exercises with the only amount of information provided in their formulation. I.e., try to understand the question, to identify what the involved notions from class are, to structure a potentially successful solving strategy. Go through these additional hints only if you get completely stuck in your first attempts.

Hints for Exercise 17. (i) Hahn-Banach + an obvious control that the extension is normalised too. (ii) Preliminary step: show that it suffices to prove that if $A = A^* \neq \mathbb{O}$ then there is a state ω with $\omega(A) \neq 0$ (for two distinct $B_1, B_2 \in \mathcal{A}$ decompose $B_1 - B_2 = C + \mathrm{i}D$, with C, D self-adjoint, and show that $\omega(B_1) \neq \omega(B_2)$ iff either $\omega(C) \neq 0$ or $\omega(D) \neq 0$). Then consider $\mathcal{A}_A \cong C(X)$, the C^*-algebra generated by A (Exercise 8): since $A \neq \mathbb{O}$ then $\widehat{A}(x) \neq 0$ for some $x \in X$, where $\widehat{A}$ is the Gelfand transform of A and apply (i) to the map $\rho(B) := \widetilde{B}(x)$ on $\mathcal{A}_A$ (fix details).

Hints for Exercise 18. (i) On the subspace $\mathcal{B} := \{\alpha\mathbb{1} + \beta A^*A \,|\, \alpha, \beta \in \mathbb{C}\}$ of $\mathcal{A}$ consider the map $\phi(\alpha\mathbb{1} + \beta A^*A) := \alpha + \beta\|A\|^2$ and prove that $\|\phi\|_{\mathcal{B}^*} = \phi(\mathbb{1}) = 1$ (use the spectral radius formula). Extend ϕ to a state ω on $\mathcal{A}$ using Hahn-Banach (check details) and show that $\omega(A^*A) = \|A\|^2$. Apply Krein-Milman to the set of states satisfying $\omega(A^*A) = \|A\|^2$ so to single out an extremal point $\widehat{\omega}$ of such a set. Prove that $\widehat{\omega}$ is pure applying the definition of pure state. (ii) Consider the GNS representation associated with the pure state found in (i) and set up a chain of inequalities starting from $\|A\|^2$.

Hints for Exercise 19. (i) Two density matrices, where $\rho_1 = e^{-\beta H}$ and ρ_2 is another function of H. (ii) (Trivial) a *free* infinite quantum system, then the KMS states are the trace states, a strict subset of all states. (Semi-trivial) an infinite quantum system with each spin interacting only with an external magnetic field.

Hints for Exercise 20. Enough to show that $\|\omega\| = \omega(\mathbb{1})$ implies that ω is positive (why enough?). To this aim, show first that $A = A^* \Rightarrow \omega(A) \in \mathbb{R}$ by considering elements of the form $A + \mathrm{i}\gamma\mathbb{1}$, $\gamma \in \mathbb{R}$ and assuming that $\omega(A) = \alpha + \mathrm{i}\beta$, $\alpha, \beta \in \mathbb{R}$. Indeed you should see that $|\omega(A + \mathrm{i}\gamma\mathbb{1})| \geqslant |\beta + \gamma\omega(\mathbb{1})|$ on the one hand, and (using the Gelfand transform) $|\omega(A + \mathrm{i}\gamma\mathbb{1})| \leqslant \omega(\mathbb{1})(\|A\|^2 + \gamma^2)^{1/2}$ on the other hand, so to deduce that necessarily (why?) $\beta = 0$. Now suppose $A \geqslant \mathbb{O}$ (without loss of generality $\|A\| \leqslant 1$, why?) and estimate the quantity $|\omega(\mathbb{1} - \omega(A)|$ in such a way to deduce $\omega(A) \geqslant \mathbb{O}$.