HOMEWORK ASSIGNMENT 04

Hand-in deadline: Wednesday 22 May 2013 by 6 p.m. in the "MSP" drop box.

Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.

Info: `www.math.lmu.de/~michel/SS13_MSP.html`

Exercise 13. Let $\mathcal{A}_{\mathrm{CAR}}(\mathfrak{h})$ be the CAR algebra over a Hilbert space $\mathfrak{h}$ and let $\mathcal{I}$ be a net of closed non-empty subspaces of $\mathfrak{h}$ ordered by inclusion such that

(1) if $M \in \mathcal{I}$ then $\exists\, N \in \mathcal{I}$ such that $M \perp N$ (in the sense of the scalar product in $\mathfrak{h}$),

(2) if $M \perp N$ and $M \perp K$, then $\exists\, L \in \mathcal{I}$ such that $M \perp L$ and $N, K \subset L$

(3) $\mathfrak{h} = \overline{\bigcup_{M \in \mathcal{I}} M}^{\|\,\|}$.

For each $M \in \mathcal{I}$ let $\mathcal{A}_M \subset \mathcal{A}_{\mathrm{CAR}}(\mathfrak{h})$ be the sub-C^*-algebra generated by $\{a(f) \,|\, f \in M\}$. Prove that $(\mathcal{A}_{\mathrm{CAR}}(\mathfrak{h}), \{\mathcal{A}_M\}_{M \in \mathcal{I}})$ is a quasi-local algebra with involutive automorphism σ such that $\sigma(a(f)) = -a(f)$ for all $f \in \mathfrak{h}$.

(*Hint:* a density argument to consider only polynomials in the $a(f)$'s and $a^*(g)$'s with f, g in each M. Consider even and odd polynomials separately. Note that A^2 and B commute if A and B anti-commute.)

Exercise 14. Let $\mathcal{A}$ be a unital C^*-algebra and let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}$.

(i) Prove that the following three conditions are equivalent:

  (1) $\ker \pi = \{\mathbb{O}\}$,

  (2) $\|\pi(A)\| = \|A\|$ for all $A \in \mathcal{A}$,

  (3) $\pi(A) > \mathbb{O}$ for all $A > \mathbb{O}$.

(ii) Assume that $\mathcal{A} = \mathcal{M}_{2\times 2}(\mathbb{C})$ and let $A = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$. Does there exist a representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$ such that $\|\pi(A)\| = \frac{1}{2}$? Give $(\mathcal{H}, \pi)$ explicitly or proof that it cannot exist.

Exercise 15. Consider the Schrödinger representation of the Weyl C^*-algebra $\mathcal{A}_W$ on $\mathbb{R}$, namely, the representation $(\mathcal{H}, \pi_S)$ where $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)$ and where the Weyl operators are realised as the unitary operators $\{U(a)\}_{a\in\mathbb{R}}$ and $\{V(b)\}_{b\in\mathbb{R}}$, with $(U(a)\psi)(x) := e^{\mathrm{i}ax}\psi(x)$ and $(V(b)\psi)(x) := \psi(x+b)$ for every $\psi \in \mathcal{H}$. Prove that $(\mathcal{H}, \pi_S)$ is irreducible.

Exercise 16. Consider the C^*-algebra $C([0,1])$ (with the $\| \|_{\text{sup}}$ norm). For each $f \in C([0,1])$ define $\omega(f) := \int_0^1 f(t)\mathrm{d}t$.

(i) Show that ω is a state on $C([0,1])$.

(ii) Find the GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ associated with the state ω, i.e., describe explicitly the Hilbert space $\mathcal{H}_\omega$, the $*$-homomorphism $\pi_\omega : C([0,1]) \to \mathcal{L}(\mathcal{H}_\omega)$, and the cyclic vector $\Omega_\omega \in \mathcal{H}_\omega$.

(iii) Prove that $\|\pi_\omega(f)\| = \|f\|_{\text{sup}}$ for all $f \in C([0,1])$.

Hints

Recommendation: try first to solve the exercises with the only amount of information provided in their formulation. I.e., try to understand the question, to identify what the involved notions from class are, to structure a potentially successful solving strategy. Go through these additional hints only if you get completely stuck in your first attempts.

Hints for Exercise 13. Direct check that $M_1 \subset M_2 \Rightarrow \mathcal{A}_{M_1} \subset \mathcal{A}_{M_2}$. The union of the local algebras is dense in $\mathcal{A}_{\mathrm{CAR}}(\mathfrak{h})$ because of assumption (3), the relation $\|a(f)\| = \|f\|$, and the fact that any $A \in \mathcal{A}_{\mathrm{CAR}}(\mathfrak{h})$ can be approximated by finite polynomials in $a(f)$ and $a^*(g)$ (check details). The common identity is... Let $\sigma : \mathcal{A}_{\mathrm{CAR}}(\mathfrak{h}) \to \mathcal{A}_{\mathrm{CAR}}(\mathfrak{h})$ be the unique $*$-automorphism such that $\sigma(a(f)) = -a(f)\ \forall f \in \mathfrak{h}$. (Existence and uniqueness of σ was only briefly sketched in class, see Theorem 5.2.5 from Bratteli and Robinson for a complete discussion.) Then $\sigma^2 =$... Then note that it suffices to prove the commutation relations for the $\mathcal{A}_M$'s for polynomials in the $a(f)$'s and $a^*(g)$'s with $f, g \in M$: to this aim, distinguish among A even and A odd with respect to σ (check details). Last, prove the commutation relations for polynomials: to this aim note that A^2 and B commute if A and B anti-commute, then use the CAR's.

Hints for Exercise 14. (i) For (1) $\Rightarrow$ (2) consider the inverse morphism π^{-1} and use the fact that π^{-1} too reduces the norm. For (2) $\Rightarrow$ (3) use the property that π maps positive elements of the C^*-algebra into positive operators on $\mathcal{H}$, and the fact that $\pi(A) \neq \mathbb{O}$ if $A > \mathbb{O}$ (why?) For (3) $\Rightarrow$ (1) go for a contradiction and take a nonzero $B \in \ker \pi$ and show that necessarily $B^*B > \mathbb{O}$, a contradiction. (ii) Consider $\ker \pi$, which is an ideal of $\mathcal{A}$ (why?). Then use (i).

Hints for Exercise 15. If $(\mathcal{H}, \pi_S)$ was not irreducible, there would exist an invariant proper subspace $\mathcal{H}_1 \subset \mathcal{H}$ and a non-zero $\phi \in \mathcal{H}_1^\perp$ (why?). Then, given $\psi \in \mathcal{H}_1$, $\langle \phi, U(a)V(b)\psi \rangle = 0$ for any $a, b \in \mathbb{R}$ (why?). Deduce that the functions $\phi(x)$ and $\psi(x+b)$ have disjoint support. And use the arbitrariness of b to conclude $\phi \equiv 0$. Contradiction.

Hints for Exercise 16. (i) Just check all properties of a state. (ii) First way: you make an educated guess, exhibiting a Hilbert space $\mathcal{H}$, a $*$-homomorphism $\pi : C([0,1]) \to \mathcal{L}(\mathcal{H})$, and the cyclic vector $\Omega \in \mathcal{H}$, such that $\omega(f) = \langle \Omega, \pi(f), \Omega \rangle_{\mathcal{H}}\ \forall f \in C([0,1])$, then you appeal to the theorem that says that the GNS is unique up to isomorphism, so you really found it. Second way: you follow step by step the constructive recipe of the GNS theorem (a bit of care in managing equivalent classes). Result: $\pi_\omega(f)$ acts as a multiplication operator... (iii) Once you solved (ii), it is a direct check.