HOMEWORK ASSIGNMENT 03
Hand-in deadline: Tuesday 14 May 2013 by 4 p.m. in the "MSP" drop box.
Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/SS13_MSP.html`

Exercise 9. Consider the fermionic Fock space $\mathfrak{F}_-(\mathfrak{h})$ built on a Hilbert space $\mathfrak{h}$ and the corresponding annihilation and creation operators $a(f)$ and $a^\dagger(f)$. Denote by Ω the vacuum of $\mathfrak{F}_-(\mathfrak{h})$. (You may find a concise recap of this construction in the note posted online in the homework/tutorials webpage.)

(i) Prove that $\|a(f)\| = \|a^\dagger(f)\| = \|f\|$ for all $f \in \mathfrak{h}$.

(ii) Let T be a bounded linear operator on $\mathfrak{F}_-(\mathfrak{h})$ that commutes with both $a(f)$ and $a^\dagger(f)$. Prove that $T = \langle \Omega, T\Omega \rangle \mathbb{1}$.

The purpose of Exercises 10 and 11 is to discuss a representation of the anti-commutation relation $U(a)V(b) = V(b)U(a)e^{-\mathrm{i}ab}$ that is different from the known Schrödinger representation where $U(a) = e^{\mathrm{i}aQ}$ and $V(b) = e^{\mathrm{i}bP}$, with Q and P respectively position and momentum operators acting on the Hilbert space $L^2(\mathbb{R})$.

Exercise 10. Consider the vector space $\mathcal{M}(\mathbb{R}) = \{\text{measurable functions } \mathbb{R} \to \mathbb{C}\}$ and the two families $\{U(a)\}_{a\in\mathbb{R}}$, $\{V(b)\}_{b\in\mathbb{R}}$ of maps on $\mathcal{M}(\mathbb{R})$ defined by

$$f \overset{U(a)}{\longmapsto} e^{\mathrm{i}ax} f\,, \qquad (V(b)f)(x) \;:=\; f(x-b)\,.$$

(i) Let $X \subset \mathcal{M}(\mathbb{R})$ be the linear span (i.e., *finite* linear combinations) of functions of the form $U(a)\mathbf{1}$ and of the form $V(b)\mathbf{1}$, with $a, b \in \mathbb{R}$ and $\mathbf{1}(x) = 1\ \forall x \in \mathbb{R}$. Explicitly,

$$X \;:=\; \Big\{ f : \mathbb{R} \to \mathbb{C} \text{ of the form } f(x) = \sum_{k=1}^{n} c_k\, e^{i\alpha_k x} \,\Big|\, n \in \mathbb{N}\,, c_k \in \mathbb{C}\,,\ \substack{\alpha_k \in \mathbb{R} \\ \text{all distinct}} \Big\}.$$

Prove that it is possible to define a scalar product $\langle\ ,\ \rangle$ on X with respect to which $U(a)$ and $V(b)$ are unitary operators for any $a, b \in \mathbb{R}$ and $\langle \mathbf{1}, \mathbf{1} \rangle = 1$. Give an explicit expression for $\langle f, g \rangle$ for $f, g \in X$.

(Recall that *unitary* means norm-preserving *and* surjective.)

(ii) Prove that $U(a)V(b) = V(b)U(a)e^{\mathrm{i}ab}\ \forall a, b \in \mathbb{R}$.

(iii) Prove that the inner product space $(X, \langle\ ,\ \rangle)$ is not complete.

Exercise 11. (Follow-up to Exercise 10.)

(i) Find the completion $\mathcal{H}$ of the non-complete inner product space $(X, \langle\ ,\ \rangle)$.

(*Note:* the completion is given by a standard construction by means of equivalence classes in X; here you are requested to make an educated guess and to identify $\mathcal{H}$ as an explicit functional space, i.e., $\mathcal{H}$ is the space of functions $\mathbb{R} \to \mathbb{C}$ such that ... and with scalar product ..., and then to recognise that X is dense in $\mathcal{H}$.)

(ii) Prove that the Hilbert space $\mathcal{H}$ thus obtained is not separable.

(*Note:* this is a first difference with respect to the separable Hilbert space $L^2(\mathbb{R})$ used for the Scrödinger representation.)

(iii) Denote again by $U(a)$ and $V(b)$ the unitary operators extended to the completion $\mathcal{H}$ of X. Is $a \mapsto U(a)$ a strongly continuous group (with respect to the strong operator topology on $\mathcal{L}(\mathcal{H})$)? Justify your answer.

(*Note:* this is the second major difference with respect to the Schrödinger representation!)

Exercise 12. Let $\mathcal{A}$ be a unital C^*-algebra. In this exercise you may use the fact that if $A \in \mathcal{A}$ is self-adjoint (i.e., $A = A^*$) then $\sigma(A) \in [-\|A\|, \|A\|]$ (see additional Problem 9(iii)).

Let $A \in \mathcal{A}$ be self-adjoint.

(i) Prove that $\|A - \mathbb{1}\| \leqslant 1 \Rightarrow A \geqslant \mathbb{O}$.

(ii) Prove that $\begin{cases} A \geqslant \mathbb{O} \\ \|A\| \leqslant 1 \end{cases} \quad \Rightarrow \quad \|A - \mathbb{1}\| \leqslant 1.$

(iii) Prove that $A \geqslant \mathbb{O} \Leftrightarrow \big\|\mathbb{1} - A/\|A\|\big\| \leqslant 1$.

Let $\mathcal{A}_+ := \{A \in \mathcal{A} \,|\, A \geqslant \mathbb{O}\}$.

(iv) Prove that $\mathcal{A}_+$ is a closed and convex cone in $\mathcal{A}$.

Note that this means, in particular, that if $A \geqslant \mathbb{O}$ and $B \geqslant \mathbb{O}$ then $A + B \geqslant \mathbb{O}$, a fact that will be often used in class.

(*Hint:* part (i) to prove convexity; to prove closedness, prove and use the fact that the self-adjoint elements of $\mathcal{A}$ are closed and the fact (following from (iii)) that $\mathcal{A}_+$ is a closed subset of the self-adjoint elements of $\mathcal{A}$.)

Hints

Recommendation: try first to solve the exercises with the only amount of information provided in their formulation. I.e., try to understand the question, to identify what the involved notions from class are, to structure a potentially successful solving strategy. Go through these additional hints only if you get completely stuck in your first attempts.

Hints for Exercise 9. (i) Compute $(a^*(f)a(f))^2$ plugging in the canonical anti-commutation relations. Use this result, and the C^*-condition, to compute $\|a(f)\|^4$. Note the subtlety: $\| \; \|$ is the norm on the CAR-algebra or on $\mathcal{L}(\mathfrak{F}_-(\mathfrak{h}))$...? (ii) Let $\{f_\alpha\}_\alpha$ be an orthonormal basis of $\mathfrak{h}$. Compute the expectation of T between two vectors of $\mathfrak{F}_-(\mathfrak{h})$ of the form $a^*(f_{\alpha_1})\cdots\alpha^*(f_{\alpha_n})\Omega$ and $a^*(f_{\beta_1})\cdots\alpha^*(f_{\beta_m})\Omega$ (in which sense are they generic vectors in $\mathfrak{F}_-(\mathfrak{h})$?) and see that only the case $n=m$ gives a non-zero value.

Hints for Exercise 10. (i) Imposing (for generic $a,b\in\mathbb{R}$) $1=\langle\mathbf{1},\mathbf{1}\rangle=\langle U(a)\mathbf{1},U(a)\mathbf{1}\rangle$ and $\langle e^{\mathrm{i}\alpha x},e^{\mathrm{i}\beta x}\rangle=\langle V(b)e^{\mathrm{i}\alpha x},V(b)e^{\mathrm{i}\beta x}\rangle$ yields $\langle e^{\mathrm{i}\alpha x},e^{\mathrm{i}\beta x}\rangle=\delta_{\alpha,\beta}$. This product extends by linearity to the whole X. Then check that $\langle f,f\rangle=0$ implies $f=0$. (ii) Direct check. (iii) The sequence $\{f_n\}_{n=1}^\infty$ in X, $f_n=\sum_{k=1}^n c_k e^{\mathrm{i}kx}$, where $c_k\neq 0$ $\forall k\in\mathbb{N}$ and $\sum_{k=1}^\infty |c_k|^2<\infty$ is a Cauchy sequence in $(X,\langle\;,\;\rangle)$ (check!), yet it does not converge when $n\to\infty$ (why?).

Hints for Exercise 11. (i) $\mathcal{H}$ consists of infinite sums whose coefficients satisfy $\sum_{k=1}^\infty |c_k|^2<\infty$. (ii) $\{e^{\mathrm{i}sx}\}_{s\in\mathbb{R}}$ is an uncountable orthonormal basis of $\mathcal{H}$, thus... (iii) Compute the quantity $\|(U(a)-\mathbb{1})e^{\mathrm{i}\alpha x}\|$ for $a\neq 0$ and deduce.

Hints for Exercise 12. (i)-(ii)-(iii) Consider the unital commutative C^* algebra $\mathcal{A}_A$ generated by A (see Exercise 8) and the corresponding Gefand isomorphism $\mathcal{A}_A \xrightarrow[\cong]{\Gamma} C(X)$. Then $A\geqslant\mathbb{O}$ if and only if $\widehat{A}\equiv\Gamma(A)$ is a positive function of X (why?). Moreover $\widehat{A}$ is a real-valued function because A is self-adjoint. Use these remarks, and the properties of Γ, to replace $\|A-\mathbb{1}\|$ with $\|\widehat{A}-\widehat{\mathbb{1}}\|$, etc., and deduce. (iv) Cone property is the check that $A\geqslant\mathbb{O}$ and $\lambda>0\Rightarrow A\geqslant\mathbb{O}$. To prove convexity, the non-trivial case is when $A\neq\mathbb{O}$ and $B\neq\mathbb{O}$, in which case apply part (i) to the element $\frac{\lambda A+\mu B}{\|A\|+\|B\|}$, where $\lambda,\mu\geqslant 0$ and $\lambda+\mu=1$. To prove closedness, prove and use the fact that the self-adjoint elements of $\mathcal{A}$ are closed in $\mathcal{A}$ and the fact (that follows from (iii)) that $\mathcal{A}_+$ is a closed subset of the self-adjoint elements of $\mathcal{A}$.