HOMEWORK ASSIGNMENT 02
Hand-in deadline: Tuesday 7 May 2013 by 4 p.m. in the "MSP" drop box.
Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/SS13_MSP.html`

Exercise 5. Let $\mathcal{A}$ be a unital Banach algebra. Assume that every element $A \in \mathcal{A}$ satisfies the identity $\|A^2\| = \|A\|^2$.

(i) Let $A \in \mathcal{A}$ be such that $\|A^2\| = \|A\|^2$. Prove that the spectral radius $r(A) := \sup\limits_{\lambda \in \sigma(A)} |\lambda|$ is such that $r(A) = \|A\|$.

(ii) Let $z \in \mathbb{C}$ and $A, B \in \mathcal{A}$, arbitrary. Prove that $\sigma(e^{-zA}Be^{zA}) = \sigma(B)$.
(The element $e^{zA} \in \mathcal{A}$ is defined by its norm-convergent series.)

(iii) Let $A, B \in \mathcal{A}$, arbitrary. Prove that $\|e^{-zA}Be^{zA}\|$ is constant for every $z \in \mathbb{C}$.

(iv) Deduce from (i)-(iii) above that the algebra $\mathcal{A}$ is necessarily commutative.
(*Hint:* $\mathbb{C} \ni z \mapsto e^{-zA}Be^{zA}$ is holomorphic.)

Exercise 6. Consider the sub-algebra $\mathcal{A}$ of $\mathcal{M}_2(\mathbb{C})$ consisting of those elements of the form $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ with $\alpha, \beta \in \mathbb{C}$.

(i) Prove that $\mathcal{A}$ is a two-dimensional commutative Banach algebra with unit $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(ii) Prove that the Gelfand transform $\Gamma : \mathcal{A} \to C(M(\mathcal{A}))$ is not injective.
(*Hint:* note that $\mathcal{A}$ is an example of unital Banach algebra that contains nilpotent elements.)

(iii) Characterise the space $C(M(\mathcal{A}))$, i.e., find all non-trivial (bounded linear) multiplicative functionals m on $\mathcal{A}$ and give the explicit action of each m on any $A \in \mathcal{A}$.

Exercise 7. Let $\mathcal{A}$ be a commutative unital Banach algebra and let $M(\mathcal{A})$ the space of all non-zero multiplicative (bounded linear) functionals on $\mathcal{A}$. Prove that $M(\mathcal{A})$ is closed in the weak-$*$ topology inherited by the topological dual $\mathcal{A}'$ of $\mathcal{A}$.

Exercise 8. Let $\mathcal{A}$ be a unital C^*-algebra. Let $A \in \mathcal{A}$ be such that $AA^* = A^*A$. Let $\mathcal{A}_A$ the unital commutative C^*-subalgebra of $\mathcal{A}$ obtained by taking the closure in $\mathcal{A}$ of the $*$-algebra of complex polynomials in A, A^*, and $\mathbb{1}$ (the so-called "C^*-ALGEBRA GENERATED BY A). Produce an explicit isometric $*$-isomorphism $\pi : \mathcal{A}_A \to C(\sigma(A))$, where $C(\sigma(A))$ is the C^*-algebra of continuous functions on the spectrum of A, such that $\pi(p(A)) = p$ for every polynomial p.

Hints

Recommendation: try first to solve the exercises with the only amount of information provided in their formulation. I.e., try to understand the question, to identify what the involved notions from class are, to structure a potentially successful solving strategy. Go through these additional hints only if you get completely stuck in your first attempts.

Hints for Exercise 5. (i) Apply the spectral radius formula $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$. (ii) First, an easy (longish, though) check that e^{zA} has the expected multiplicative properties, in particular it is invertible. Then note that $(\lambda - e^{-zA}Be^{zA}) = e^{-zA}(\lambda - B)e^{zA}$ and discuss the non-invertibility of both sides. (iii) From (i) $\|e^{-zA}Be^{zA}\| = r(e^{-zA}Be^{zA})$ and from (ii) $r(e^{-zA}Be^{zA}) = r(B)$. (iv) The function $z \mapsto e^{-zA}Be^{zA}$ being holomorphic (with values in $\mathcal{A}$) and bounded, necessarily it is constant. From $e^{-zA}Be^{zA} = B - z(AB - BA) + \dots$ and from the arbitrariness of z, it follows $AB = BA$.

Hints for Exercise 6. (i) The basis consists of two obvious elements... All properties of commutative algebra are also readily checked. A cheap way to prove that $\mathcal{A}$ is a Banach algebra is to show that it is a closed subspace of the Banach algebra $\mathcal{M}_2(\mathbb{C})$. (The norm/algebraic relations are inherited. As for completeness you use the fact that a closed subset of a complete space is complete.) (ii) The non-zero A's in $\mathcal{A}$ on which the Gelfand transform Γ fails to be injective (i.e., $\Gamma(A) = 0$) turn out to be those such that $\|A^n\|^{1/n} \xrightarrow{n\to\infty} 0$ ("quasi-nilpotent" elements). Why...? (Answer: $\Gamma(A) = 0 \Rightarrow m(A) = 0$ for every multiplicative functional on $\mathcal{A}$ $\Rightarrow \sigma(A) = \{0\} \Rightarrow$ spectral radius of A is zero, and spectral radius $= \lim_{n\to\infty} \|A^n\|^{1/n}$. Justify these easy steps.) But $\mathcal{A}$ contains such elements, therefore Γ is not injective. For instance, $Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ... (iii) If m is a non-zero multiplicative functional onf $\mathcal{A}$, then $m(\mathbb{1}) = 1$ and $m(Q) = 0$, where $Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Why...? (Answer: $Q^2 = \mathbb{O}$, so...). This gives the action of m on the generic element $\alpha\mathbb{1} + \beta Q$ of $\mathcal{A}$. Which also says that there is only one non-zero m.

Hints for Exercise 7. Remark: consider the special case when $\mathcal{A}$ is separable; then the unit ball in the dual space is metrizable in the weak-$*$ topology (a fact you might have encountered in some previous courses), therefore for $M(\mathcal{A})$ the property of being closed is equivalent to being sequentially closed, i.e., you can check that $M(\mathcal{A})$ is closed using sequences. In the general case, namely when $\mathcal{A}$ is not separable, you have to make use of *nets* ("generalised sequences") and to exploit the theorem that says that a set is closed iff it contains the limit points of all its nets. Recall the property revisited in tutorials: if $\{\omega_\alpha\}_{\alpha\in\mathcal{I}}$ is a net in $\mathcal{A}^*$, $\omega_\alpha \to \omega \in \mathcal{A}^*$ in the weakly-$*$ sense iff $\omega_\alpha(A) \to \omega(A)\ \forall A \in \mathcal{A}$. The exercise then consists of proving that the multiplicative property $\omega_\alpha(AB) = \omega_\alpha(A)\omega_\alpha(B)$ survives in the limit.

Hints for Exercise 8. Via the Gelfand transform Γ, $\mathcal{A}_A$ is isomorphic to $C(X)$, where X is the compact topological space consisting of the characters of $\mathcal{A}_A$ equipped with the weak-$*$ topology of $\mathcal{A}_A^*$. On the other hand, $\Gamma(A) : X \to \sigma(A)$ is a homeomorphism (why?). Define $\alpha : C(X) \to C(\sigma(A))$ by $\alpha(f) = f \circ \Gamma(A)^{-1}$ (in particular: $(\alpha(\Gamma(A)))(\lambda) = \lambda\ \forall\lambda \in \sigma(A)$). Then α is an isometric $*$-isomorphism (why?). It turns out that $\pi := \alpha \circ \Gamma$ is the requested isometric $*$-isomorphism (check!).