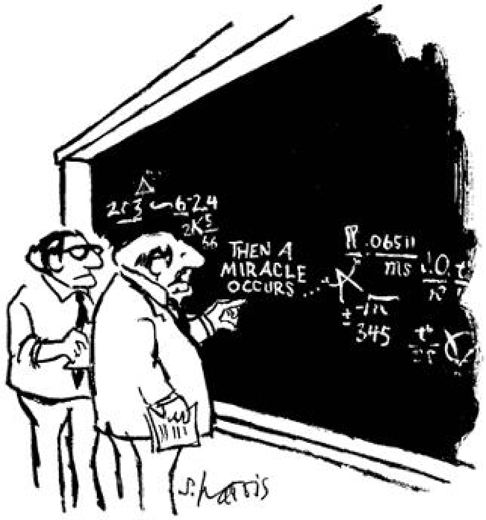
Mathematical Statistical Physics

TMP Programme Munich – spring term 2013

HOMEWORK ASSIGNMENT 01
Hand-in deadline: Tuesday 30 April 2013 by 4 p.m. in the "MSP" drop box.
Rules: Each exercise is worth 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions in German or in English.
Info: `www.math.lmu.de/~michel/SS13_MSP.html`



"I think you should be more explicit here in step two."

Exercise 1. Consider the Banach space $\ell^1(\mathbb{Z})$ with its natural norm $\| \, \|_1$. (Recall: $\ell^1(\mathbb{Z}) = \{x \equiv (x_n)_{n\in\mathbb{Z}} \,|\, x_n \in \mathbb{C} \text{ and } \|x\|_1 := \sum_{n\in\mathbb{Z}} |x_n| < +\infty\}$.) For every $x, y \in \ell^1(\mathbb{Z})$ define

$$x * y \equiv \big((x*y)_n\big)_{n\in\mathbb{Z}}\,, \qquad (x*y)_n := \sum_{m\in\mathbb{Z}} x_m y_{n-m}\,,$$

and

$$x^* \equiv (x_n^*)_{n\in\mathbb{Z}}\,, \qquad x_n^* := \overline{x_{-n}}\,.$$

(i) Prove that $x * y$ is an internal product and x^* is an involution that make $\ell^1(\mathbb{Z})$ a commutative Banach-$*$ algebra.

(ii) Is the Banach-$*$ algebra $\ell^1(\mathbb{Z})$ also a C^*-algebra? Justify your answer.

Exercise 2. Prove that a commutative and unital Banach-algebra $\mathcal{A}$ whose only closed ideals are the trivial ones (i.e., $\{\mathbb{O}\}$ and $\mathcal{A}$ itself) is necessarily $\mathcal{A} \cong \mathbb{C}$.

Hint: assume that there exists a non-scalar $A \in \mathcal{A}$ and take $\lambda \in \sigma(A)$. Then construct a convenient closed ideal.

Exercise 3. Let $\mathcal{A}$ be a C^*-algebra without unit. Consider the collection $\widetilde{\mathcal{A}}$ of pairs (A, λ), where $A \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, and equip it with the operations

$$\begin{aligned} (A,\lambda) + (B,\mu) &:= (A+B, \lambda+\mu) \\ \mu(A,\lambda) &:= (\mu A, \mu\lambda) \\ (A,\lambda)(B,\mu) &:= (AB + \lambda B + \mu A, \lambda\mu) \\ (A,\lambda)^* &:= (A^*, \overline{\lambda})\,. \end{aligned} \tag{1}$$

Define

$$\|(A,\lambda)\| \;:=\; \sup_{\substack{B\in\mathcal{A},\\ \|B\|=1}} \|AB + \lambda B\|\,. \tag{2}$$

(i) Prove that $\widetilde{\mathcal{A}}$ with the operations (1) is a unital algebra with an involution.

(ii) Prove that (2) defines a norm that makes $\widetilde{\mathcal{A}}$ a C^*-algebra into which $\mathcal{A}$ is embedded via the isometric embedding $A \mapsto (A, 0)$.

(Comment: the *non-unital* $\mathcal{A}$ is therefore naturally contained in a *unital* C^*-algebra $\widetilde{\mathcal{A}}$, via the identification $A \equiv (A, 0)$, as a maximal ideal of co-dimension one.)

(iii) Consider the concrete case $\mathcal{A} = C_\infty(\mathbb{R})$, the space of $\mathbb{R} \to \mathbb{C}$ continuous functions vanishing at infinity, with the supremum norm. Which classical functional space can then $\widetilde{\mathcal{A}}$ be identified with?

Exercise 4. Consider the interval $J = [-2013, 2013]$, the commutative unital C^*-algebra $C(J)$, and the element $f \in C(J)$, $f(x) := \arctan \frac{x}{x^2+1}$. Compute $\sigma(f)$, the spectrum of f. Justify your answer.

(Note: as discussed in class, $C(J)$ is meant to be equipped with the usual point-wise product, complex conjugation, and supremum norm.)

Hints

Recommendation: try first to solve the exercises with the only amount of information provided in their formulation. I.e., try to understand the question, to identify what the involved notions from class are, to structure a potentially successful solving strategy. Go through these additional hints only if you get completely stuck in your first attempts.

Hints for Exercise 1. (i) is just a check of all properties of the definition from class. Note that the product is commutative because... In (ii) you should argue that the C^*-condition $\|x^*x\|_1 = \|x\|_1^2$ fails to hold for some $x \in \ell^1(\mathbb{Z})$. For instance design x to be conveniently "peaked" at the origin, i.e., only $x_{-1}, x_0, x_1 \neq 0$, say $x_{-1} = x_1$. Then the C^*-condition cannot work.

Hints for Exercise 2. As suggested, take $A \neq \alpha\mathbb{1}$ in the algebra $\mathcal{A}$ (where $\alpha \in \mathbb{C} \setminus \{0\}$) and $\lambda \in \sigma(A)$. Consider the ideal $\mathcal{I} = \overline{(A - \lambda\mathbb{1})\mathcal{A}}$, that is closed by construction. The strategy is to come to the contradiction that $\mathcal{I}$ is a *proper* ideal, so that $\mathcal{A}$ necessarily consists of only scalar A's. To this aim, it is enough to produce one element of $\mathcal{A}$ that does not belong to $\mathcal{I}$. A natural candidate is... the identity $\mathbb{1}$ (why natural?). Therefore the strategy is to show that $\mathbb{1}$ cannot be approximated in norm by elements of the form $(A - \lambda\mathbb{1})B$, $B \in \mathcal{A}$. Equivalently, the strategy is to show that $\|(A - \lambda\mathbb{1})B - \mathbb{1}\|$ cannot be arbitrarily small when B runs over $\mathcal{A}$. In fact, you should now be able to see that $\|(A - \lambda\mathbb{1})B - \mathbb{1}\| \geqslant 1\ \forall B \in \mathcal{A}$. Why? Suggestion: no element of the form $(A - \lambda\mathbb{1})B$ is invertible, because λ is in the spectrum of A. Thus...

Hints for Exercise 3. (i) is just a check of all properties of the definition from class. (ii) includes standard checks too. Triangle and product inequality for $\|(A, \lambda)\|$ are standard. For the check that $\|(A, \lambda)\| = 0$ implies $A = \mathbb{O}$ and $\lambda = 0$ it is enough to prove that $\|(A, 1)\| \neq 0$ (why?). To this aim, use $\|AB - B\| \leqslant \|B\|\,\|(-A, 1)\|$: this tells you that $\|(-A, 1)\| = 0 \Rightarrow AB = B\ \forall B \in \mathcal{A}$. Using the involution in $\mathcal{A}$ you should be able to deduce that A is an identity, a contradiction. Alternatively, you can prove that the norm $\|(A, \lambda)\|$ is a Banach algebra norm using an abstract argument: $\|(A, \lambda)\|$ is the norm induced from the space $\mathcal{B}(\mathcal{A})$ of bounded operators on $\mathcal{A}$ given by the $*$-algebra of operators $\|L_A + \lambda\mathbb{1} \,|\, A \in \mathcal{A}, \lambda \in \mathbb{C}\}$, where $L_A(B) := AB$ (check all details!). The embedding $A \mapsto (A, 0)$ is isometric because $\|A\| = \|A(A^*/\|A\|)\| \leqslant \|(A, 0)\| \leqslant \|A\|$ (why...?), whence $\|A\| = \|(A, 0)\|$. At this point you are left with the check of the C^*-condition, which you should prove showing first that $\|(A, \lambda)\| \leqslant \|(A, \lambda)^*\|$ and then exchanging the role of (A, λ) and $(A, \lambda)^*$.

Hints for Exercise 4. The general fact you should justify is that the spectrum of an element in $C(J)$ is the range of that function. This follows from the definition of spectrum. The rest is high school Calculus (therefore the part you are most likely to fail...).