Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
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PROBLEM IN CLASS – WEEK 10

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/SS12_FA.html`.

Problem 37. (A continuity property for a sequence of orthogonal projections)

Let $\mathcal{H}$ be a Hilbert space, $V_1 \supset V_2 \supset \cdots \supset V_n \supset V_{n+1} \supset \cdots$ be a sequence of closed linear subspaces, $V := \cap_{n\in\mathbb{N}} V_n$, and $P_n : \mathcal{H} \to \mathcal{H}$, resp. $P : \mathcal{H} \to \mathcal{H}$, be the orthogonal projection operator onto V_n, resp. V. Show that $\lim_{n\to\infty} P_n x = Px \ \forall x \in \mathcal{H}$.

Problem 38. (An orthogonal projection on $L^2(\mathbb{R}^+)$)

Let $V := \Big\{ f \in L^2(\mathbb{R}^+) \,\Big|\, \displaystyle\int_0^n f(x)\,\mathrm{d}x = 0 \ \ \forall n \in \mathbb{N} \Big\}$. Show that V is a closed linear subspace of $L^2(\mathbb{R}^+)$ and find the orthogonal projection $P_V : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ onto V.

Problem 39. (Solutions to the initial value problem of a first-order ODE: existence and uniqueness à la Peano and à la Picard.)

The following are given: $d \in \mathbb{N}$, $t_0 \in \mathbb{R}$, $u_0 \in \mathbb{R}^d$, a neighbourhood $\Omega \ni (t_0, u_0)$ open in $\mathbb{R}^{d+1}$, $f : \Omega \to \mathbb{R}^d$. Consider the initial value problem ($\dot{u} \equiv \frac{\mathrm{d}u}{\mathrm{d}t}$)

$$\begin{cases} \dot{u} = f(t,u) \\ u(0) = u_0 \,. \end{cases} \tag{$*$}$$

(i) (Peano existence theorem.) Assume that f is continuous. Show that there exists at least one solution $u \in C^1([t_0-\delta, t_0+\delta], \mathbb{R}^d)$ to $(*)$ for some $\delta > 0$ sufficiently small.

(ii) (Picard-Lindelöf existence and uniqueness theorem.) Assume that f is continuous and that $f(t,\cdot)$ is Lipschitz uniformly in t. Show that there exists a unique solution $u \in C^1([t_0-\delta, t_0+\delta], \mathbb{R}^d)$ to $(*)$ for some $\delta > 0$ sufficiently small.

Problem 40. (Smooth functions are not dense in Hölder-continuous functions)

Let $\alpha \in (0,1]$. Show that $C^1([0,1]) \subset C^{0,\alpha}([0,1])$ but the inclusion is not dense.