Institute of Mathematics, LMU Munich – Spring Term 2012
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PROBLEM IN CLASS – WEEK 7

These additional problems are for your own preparation at home. They supplement examples and properties not discussed in class. Some of them will discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for the final exam. Further info at `www.math.lmu.de/~michel/SS12_FA.html`.

Problem 25. (Equality in Minkowski's inequality. Variational characterization of norm in ℓ^p spaces and in Hilbert spaces.)

(i) Consider the triangular (Minkowski) inequality $\|x+y\|_p \leqslant \|x\|_p + \|y\|_p$, valid for all $x, y \in \ell^p$, $1 \leqslant p \leqslant \infty$ (Example 2.4(3) from class). Show that if $1 < p < \infty$ this inequality takes the "=" sign if and only if x and y are linearly dependent, i.e.,

$$\|x+y\|_p = \|x\|_p + \|y\|_p \qquad \Leftrightarrow \qquad y = \alpha x \quad \text{for some } \alpha \geqslant 0 \qquad (x \neq 0,\ p \in (0,1)),$$

whereas if $p = 1$ or $p = \infty$ equality can hold in other cases as well.

(*Hint:* revisit the derivation of Minkowski's inequality from Hölder's ($\sum_n |a_n b_n| \leqslant \|a\|_p \|b\|_{p'}$) and use the fact that equality in Hölder holds if and only if $|a_n| = \lambda |b_n|^{p-1}$ $\forall n \in \mathbb{N}$ for some $\lambda \geqslant 0$.)

(ii) Let $p \in [1, \infty)$ and let q be the Hölder dual index to p. Show that for any $x \in \ell^p$

$$\|x\|_p \;=\; \sup_{\substack{y \in \ell^q \\ \|y\|_q = 1}} \Big| \sum_{n=1}^{\infty} x_n y_n \Big| \qquad \text{and also} \qquad \|x\|_p \;=\; \sup_{\substack{y \in c_{00} \\ \|y\|_q = 1}} \Big| \sum_{n=1}^{\infty} x_n y_n \Big| \,.$$

(iii) Let $(\mathcal{H}, \langle\ ,\ \rangle)$ be a Hilbert space. Show that for any $x \in \mathcal{H}$

$$\|x\| \;=\; \sup_{\substack{y \in \mathcal{H} \\ \|y\| = 1}} \big| \langle y, x \rangle \big| \qquad \text{and also} \qquad \|x\| \;=\; \sup_{\substack{y \in \mathscr{D} \\ \|y\| = 1}} \big| \langle y, x \rangle \big| \,,$$

where $\mathscr{D}$ is a dense linear subspace of $\mathcal{H}$.

Problem 26. (Algebraic equation for compact operators. The l^p spaces are nested.)

(i) The following are given: $p \in [1, \infty]$, $N \in \mathbb{N}$, a linear compact operator T on ℓ^p. For which scalars $a_0, a_1, \dots, a_N$ can T satisfy an algebraic equation $\displaystyle\sum_{n=0}^{N} a_n T^n = \mathbb{O}$? ($T^0 \equiv \mathbb{1}$.) Justify your answer.

(ii) Let $x \in \ell^p$ for some $p \in [1, \infty]$. Show that $x \in \ell^q$ for all $q \in [p, \infty]$ and

$$\lim_{q \to \infty} \|x\|_q \;=\; \|x\|_\infty \,.$$

(iii) Is the embedding $\ell^p \hookrightarrow \ell^q$ $(1 \leqslant p \leqslant q \leqslant \infty)$ compact? Justify your answer.

Problem 27. (Ptolemaic inequality. The unit ball is strictly convex in Hilbert spaces.)

Let $\mathcal{H}$ be an inner product space, say with norm $\| \; \|$ and scalar product $\langle \, , \, \rangle$.

(i) Let $x, y \in \mathcal{H} \setminus \{0\}$ and set $\widetilde{x} := \dfrac{x}{\|x\|^2}$, $\widetilde{y} := \dfrac{y}{\|y\|^2}$. Show that $\|\widetilde{x} - \widetilde{y}\| = \dfrac{\|x-y\|}{\|x\| \, \|y\|}$.

(ii) Show that for any $x, y, z, w \in \mathcal{H}$ one has (Ptolemaic inequality)

$$\|x - z\| \, \|y - w\| \;\leqslant\; \|x - y\| \, \|z - w\| + \|y - z\| \, \|x - w\| \, .$$

In addition, assume now that $\mathcal{H}$ is a Hilbert space.

(iii) Show that the closed unit ball in $\mathcal{H}$ is strictly convex, whereas in general the closed unit ball in a normed space is only convex.

(Recall: $K \subset \mathcal{H}$ is convex if given any $x, y \in K$ then $tx + (1-t)y \in K$ for all $t \in [0,1]$. K is strictly convex if given any $x, y \in \partial K$ the segment $\{tx + (1-t)y \,|\, t \in [0,1]\}$ intersects ∂K only in the points x and y, i.e., only for $t = 0$ or 1. Recall also that there are various equivalent characterizations of strict convexity of a closed unit ball in a normed space.)

Problem 28. (Constructing new Banach spaces as quotients of Banach spaces.)

Let $(X, \| \; \|)$ be a normed space and let $Y \subset X$ be a proper closed subspace.

(i) Define the relation $\sim$ on X by $x \sim x'$ iff $x - x' \in Y$. Show that $\sim$ is an equivalence relation.

(ii) Denote by X/Y, or equivalently $X/\!\sim$, the set of equivalence classes w.r.t. $\sim$, i.e., elements $[x] = \{x + y \,|\, y \in Y\}$. Show that $[x] + [x'] := [x + x']$ and $[\lambda x] := \lambda [x]$ $(x, x' \in X,\ \lambda \in \mathbb{C})$ give a well-defined$^{(*)}$ linear structure in X/Y.

(iii) Define $\|[x]\|_\sim := \inf_{y \in Y} \|x - y\|$. Show that this definition is well-posed (i.e., it does not depend on the choice of the representative x for the equivalence class $[x]$) and that $\| \; \|_\sim$ is a norm on X/Y.

(iv) Define the projection of X onto the quotient X/Y as the map $\pi : X \to X/Y$ such that $x \mapsto [x]$. Show that π is bounded with $\|\pi\| = 1$.

(v) Show that $(X/Y, \| \; \|_\sim)$ is complete, and therefore is a Banach space, under the additional assumption that $(X, \| \; \|)$ is a Banach space.