Functional Analysis

Institute of Mathematics, LMU Munich – Spring Term 2012
Prof. T. Ø. Sørensen Ph.D, A. Michelangeli Ph.D

HOMEWORK ASSIGNMENT no. 3, issued on Tuesday 1 May 2012
Due: Tuesday 8 May 2012 by 6 pm in the designated "FA" box on the 1st floor
Info: www.math.lmu.de/~michel/SS12_FA.html

Each exercise is worth a full mark of 10 points. Correct answers without proofs are not accepted. Each step should be justified. You can hand in your solutions either in German or in English.

Exercise 9. (Hausdorff topological spaces.)

(i) Show that a sequence in a Hausdorff topological space cannot converge to more than one point.

(ii) Show that if X is a Hausdorff topological space and $x \in X$ then $\{x\}$ is a closed set and more generally any finite subset $E \subset X$ is closed.

(iii) Converse to (ii): if each point of a topological space is a closed set, is that space necessarily a Hausdorff space? Justify your answer with a proof or a counterexample.

(iv) Show that any subset of a Hausdorff space is itself a Hausdorff space in the relative topology.

(v) Produce a topology on $B := \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leqslant 1\}$ which has uncountably many opens, is <u>different</u> from (i.e., non equivalent to) the co-finite topology (Exercise 2), the co-countable topology (Exercise 8), the either-or topology (Problem 7(i)), or the partition topology (Problem 7(ii)), and makes B a non-Hausdorff space.

Exercise 10. (Second countable implies separable, not vice versa.)

(i) Let Y be a second countable topological space. Show that Y is separable.

(ii) Show that a metric space is second countable *if and only if* it is separable.

(iii) Show that the topological space $(\mathbb{R}, \mathcal{T})$ considered in Problem 6 is separable and first countable, but *not* second countable.

(iv) Let X be a topological space equipped with the cofinite topology $\mathcal{T}$ defined in Exercise 2. Show that $(X, \mathcal{T})$ is separable.

(v) Follow-up to (iv): what cardinality has X to have in order to be second countable? Justify your answer.

(Note therefore that $(\mathbb{R}, \mathcal{T})$ in (iii) and $(X, \mathcal{T})$ in (v) are examples of separable but not second countable topological spaces. One more example is discussed in Problem 11(ii).)

Exercise 11. (Basic properties of the finite product topology.)

Let Z, X_1 and X_2 be topological spaces and let $X := X_1 \times X_2$ with the product topology. Let $\pi_j : X \to X_j$, $j = 1, 2$, be the projection onto the j-th component.

(i) Show that π_1 and π_2 are continuous, open, not necessarily closed maps.

(ii) Show that the product topology on X is the *smallest* topology for which π_1 and π_2 are continuous.

(iii) Show that a function $f : Z \to X$ is continuous if and only if each $\pi_j \circ f : Z \to X_j$ is, $j = 1, 2$.

(iv) Show that X is a Hausdorff space if and only if both X_1 and X_2 are.

(It is of course understood that each $X_j \neq \emptyset$, $j = 1, 2$.)

(v) Show that Z is a Hausdorff space if and only if the "diagonal" $\Delta := \{(z, z) \,|\, z \in Z\}$ is closed in $Z \times Z$ with respect to the product topology.

Exercise 12. (A metric on $\mathbb{R}$ making each rational an open set.)

(i) Show that $\mathbb{Q}$ is neither open nor closed in $\mathbb{R}$ with respect to the Euclidean metric.

(ii) Produce a metric d on $\mathbb{R}$ (different from the discrete metric) such that for each $q \in \mathbb{Q}$ the set $\{q\}$ is open in $(\mathbb{R}, d)$ and therefore $\mathbb{Q}$ itself is an open subset of $(\mathbb{R}, d)$.

(*Hint:* design d so that the ball $B_\varepsilon(q)$ of radius ε centred at $q \in \mathbb{Q}$ contains q only, if ε is sufficiently small.)