Advanced Mathematical Quantum Mechanics – Final Test, 30.07.2011

Fortgeschrittene Mathematische Quantenmechanik – Endklausur, 30.07.2011

Name:/*Name*: ______________________________

Matriculation number:/*Matrikelnr.:* ____________ **Semester:**/*Fachsemester:* ____________

Degree course:/*Studiengang:* ❑ Bachelor PO 2007 ❑ Lehramt Gymnasium (modularisiert)
❑ Bachelor PO 2010 ❑ Lehramt Gymnasium (nicht modularisiert)
❑ Diplom ❑ Master ❑ TMP ❑ ____________

Major:/*Hauptfach:* ❑ Mathematik ❑ Wirtschaftsm. ❑ Informatik ❑ Physik ❑ Statistik ❑ ________

Minor:/*Nebenfach:* ❑ Mathematik ❑ Wirtschaftsm. ❑ Informatik ❑ Physik ❑ Statistik ❑ ________

Credits needed for:/*Anrechnung der Credit Points für das:* ❑ Hauptfach ❑ Nebenfach (Bachelor/Master)

Extra solution sheets submitted:/*Zusätzlich abgegebene Lösungsblätter:* ❑ Yes ❑ No

problem	1	2	3	4	5	6	7	8	9	$\sum$
total points	10	15	15	15	15	15	20	20	25	150
scored points										

homework performance		final test performance		total performance		FINAL MARK	

INSTRUCTIONS:

- This booklet is made of twenty-two pages, including the cover, numbered from 1 to 22. The test consists of nine problems. Each problem is worth the number of points specified in the table above. 100 points are counted as 100% performance in this test. You are free to attempt any problem and collect partial credits.
- The only material that you are allowed to use is black or blue pens/pencils and one two-sided A4-paper "cheat sheet" (Spickzettel). You cannot use your own paper: should you need more paper, raise your hand and you will be given extra sheets.
- Prove all your statements or refer to the standard material discussed in class.
- Work individually. Write with legible handwriting. You may hand in your solution in English or in German. Put your name on every sheet you hand in.
- You have 150 minutes.

GOOD LUCK!

Fill in the form here below only if you need the certificate (Schein).

Dieser Leistungsnachweis entspricht auch den Anforderungen
nach § Abs. Nr. Buchstabe LPO I
nach § Abs. Nr. Buchstabe LPO I

UNIVERSITÄT MÜNCHEN

ZEUGNIS

Der / Die Studierende der __________

Herr / Frau __________ *aus* __________

geboren am _____ *in* _____ *hat im* **SoSe** _____ *–Halbjahr* **2011** _____

meine Übungen **zur Fortgeschritt. Mathematischen Quantenmechanik** _____

mit __________ *besucht.*

Er / Sie hat __________

schriftliche Arbeiten geliefert, die mit ihm / ihr besprochen wurden. _____

MÜNCHEN, *den* **30 Juli 2011**

Name

PROBLEM 1. (10 points) Let A be a self-adjoint operator on a given Hilbert space $\mathcal{H}$. Define the operator $U := (A + i\mathbb{1})(A - i\mathbb{1})^{-1}$.

(i) Prove that U is a unitary operator $\mathcal{H} \to \mathcal{H}$.

(i) Prove that $\mathrm{Ker}(U - \mathbb{1}) = \{0\}$.

SOLUTION:

Name

PROBLEM 2. (15 points) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

(i) Explain whether A admits a cyclic vector in $\mathbb{C}^3$ or not.

(ii) Decompose A into multiplication form with minimal spectral multiplicity.

SOLUTION:

Name

PROBLEM 3. (15 points) Let A be the multiplication operator by $G(x) \geqslant 0$ in $L^2(\mathbb{R}^d)$ on the natural domain $\mathcal{D}(A) = \{f \in L^2 | Gf \in L^2\}$. Show that the quadratic form associated with A has form domain $\mathcal{D}(q_A) = \{f \in L^2 | \sqrt{G} f \in L^2\}$ and is given by $q_A(f,g) = \int_{\mathbb{R}^d} \overline{f(x)} G(x) g(x)$. (*Hint:* check the closability.)

SOLUTION:

Name

PROBLEM 4 (15 points). Recall that ℓ^2 is the space of sequences $(x_1, x_2, x_3, \dots)$ such that each $x_n \in \mathbb{C}$ and $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. Recall also that c_{00} is the subspace of ℓ^2 of sequences with only finitely many non-zero entries. Consider the operator A on ℓ^2 with domain $\mathcal{D}(A) = c_{00}$ and action $(Ax)_n := nx_n$ $(n = 1, 2, 3, \dots)$ $\forall x \in c_{00}$.

(i) Find A^*.

(ii) Find $\overline{A}$.

(iii) Find all self-adjoint extensions of A.

(Notice: by "find an operator" one means find its domain and its action.)

SOLUTION:

Name

PROBLEM 5. (15 points) Consider the sequence $\{A_n\}_{n=1}^{\infty}$ of self-adjoint operators, densely defined on the same given Hilbert space $\mathcal{H}$, and let A be another self-adjoint operator on $\mathcal{H}$. Assume that

$$\lim_{n\to\infty} \left\| e^{itA_n}\varphi - e^{itA}\varphi \right\| = 0 \qquad \forall \varphi \in \mathcal{H}\,, \qquad \forall t \in \mathbb{R}\,.$$

Show that

$$\lim_{n\to\infty} \left\| R_z(A_n)\varphi - R_z(A)\varphi \right\| = 0 \qquad \forall \varphi \in \mathcal{H}$$

where $R_z(A_n) = (z\mathbb{1} - A_n)^{-1}$, $R_z(A) = (z\mathbb{1} - A)^{-1}$ for an arbitrary $z \in \mathbb{C}\setminus\mathbb{R}$. (*Hint:* represent the resolvent $R_z(A)$ with an integral involving e^{itA}.)

SOLUTION:

Name

PROBLEM 6. (15 points) Find all solutions $\{\lambda, f\}$, with $\lambda \in \mathbb{C}$ and $f \in L^2[0,1]$, to the integral equation

$$\int_0^1 \cos 2\pi(x-y) f(y) \mathrm{d}y = \lambda f(x) \qquad \text{a.e. } x \in \mathbb{R}\,.$$

SOLUTION:

Name

PROBLEM 7. (15 points) Consider the one-parameter group $\{U(t)\}_{t\in\mathbb{R}}$ of unitary operators on a given Hilbert space $\mathcal{H}$ such that $U(1) = \mathbb{1}$. Prove that there exists a self-adjoint operator A on $\mathcal{H}$ such that $U(t) = e^{2\pi i t A}$, $\forall t \in \mathbb{R}$, and such that $\sigma_{\mathrm{pp}}(A) \subset \mathbb{Z}$.

SOLUTION:

Name

PROBLEM 8. (20 points) Consider the right shift operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $(T\psi)(x) := \psi(x-1)$ for a.e. $x \in \mathbb{R}$ and $\forall \psi \in L^2(\mathbb{R})$.

(i) Prove that T is a unitary operator on $L^2(\mathbb{R})$.

(ii) Prove that $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. (*Hint:* one inclusion was discussed in one tutorial, but you have to prove it here. For the other inclusion use Weyl's criterion. Notice that the prototype for a suitable Weyl's sequence is the function $e^{-i\eta x}$, if $\lambda = e^{i\eta}$ for some $\eta \in [0, 2\pi)$.)

(iii) Prove that $\sigma_{\mathrm{pp}}(T) = \emptyset$.

(iv) Let $f \in L^2(\mathbb{R})$ and $z \in \rho(T)$. Give an as explicit as possible formula to compute $(z-T)^{-1}f$. (*Hint:* distinguish $|z| < 1$ and $|z| > 1$, and in both cases write the Neumann series for the resolvent).

SOLUTION:

Name

PROBLEM 9. (25 points) Consider a real-valued potential V such that $V = V_1 + V_2$ where $V_1 \in L^\infty(\mathbb{R}^3)$ and vanishes at infinity, and $V_2 \in L^2(\mathbb{R}^3)$. Let $H_0 = -\Delta$ on $\mathcal{D}(H_0) = H^2(\mathbb{R}^3)$.

(i) Show that $V : \mathcal{D}(H_0) \to L^2(\mathbb{R}^3)$ (as a multiplication operator) and thus $H_0 + V$ is well defined on $\mathcal{D}(H_0)$. (*Hint:* the same as when in MQM-1 we proved $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$.)

(ii) Consider the operator $H = H_0 + V$ with domain $\mathcal{D}(H) = H^2(\mathbb{R}^3)$. Show that H is self-adjoint. (*Hint:* Kato-Rellich.)

(iii) Show that $\sigma_{\mathrm{ess}}(H) = [0, \infty)$. (*Hint:* Use the theorem from class that $\sigma_{\mathrm{ess}}(A+B) = \sigma_{\mathrm{ess}}(A)$ if B is relatively compact with respect to A, and use the $f(x)g(\nabla)$ theorem from MQM-1.)

SOLUTION: