INDIVIDUAL PROJECT NO.3, issued on Thursday 7 July 2011
Due: Monday 25 July 2011 in the tutorial session
Info: `www.math.lmu.de/~michel/SS11_MQM2.html`

Work out individually the details of the problem outlined in the scheme below. Results and techniques discussed in the class as well as in the tutorial and exercise sessions will be needed.

Consider the mean-field model of a three-dimensional, non-relativistic system of N indistiguishable, spinless bosons of mass $m = \frac{1}{2}$ whose Hamiltonian (in units $\hbar = 1$) is

$$H_N \;=\; \sum_{j=1}^{N}(-\Delta_{x_j}) \;+\; \frac{1}{N^2}\sum_{1\leqslant i<j<k\leqslant N} W(x_i,x_j,x_k) \tag{1}$$

(each $x_i \in \mathbb{R}^3$). Here $W : \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}$ is a translation invariant three-body potential that is symmetric under permutation of particles. For the purpose of this problem W is assumed to have the particular form

$$W(x,y,z) \;=\; V(x-y)V(x-z) + V(x-y)V(y-z) + V(x-z)V(y-z) \qquad \forall\, x,y,z \in \mathbb{R}^3 \tag{2}$$

for some given $V \in L^\infty(\mathbb{R}^3)$, with compact support, and spherically symmetric, i.e., $V(x) = V(-x)$ $\forall x \in \mathbb{R}^3$. Notice that, consistently, the appropriate mean-field pre-factor is now N^{-2}, instead of N^{-1} used for the two-body interaction case discussed in class.

The goal of this problem is to prove the following:
Theorem. *Let $\Psi_{N,t}$ be the (unique) solution to the (many-body) Schrödinger equation*

$$i\partial_t\Psi_{N,t} \;=\; H_N\Psi_{N,t} \tag{3}$$

with initial data $\Psi_{N,t}\big|_{t=0} = \varphi^{\otimes N}$, where φ is a given (one-body) wave-function in $L^2(\mathbb{R}^3)$ with $\|\varphi\|_2 = 1$. Let $\gamma^{(k)}_{N,t}$, $k = 1,\dots,N$, be the k-particle reduced density matrix associated with $\Psi_{N,t}$. Then, for every $t \in \mathbb{R}$ and for any fixed k,

$$\lim_{N\to\infty} \mathrm{Tr}\,\big|\gamma^{(k)}_{N,t} - |\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}|\,\big| \;=\; 0 \tag{4}$$

where φ_t is the solution to the (one-body, "quintic") non-linear Schrödinger equation

$$i\partial_t\varphi_t \;=\; -\Delta\varphi_t \;+\; \frac{1}{2}\,(V*|\varphi_t|^2)^2\,\varphi_t \;+\; \big(V*((V*|\varphi_t|^2)|\varphi_t|^2)\big)\varphi_t \tag{5}$$

with initial data $\varphi_t\big|_{t=0} = \varphi$.

(i) Plug in (3) the ansatz $\Psi_{N,t} = \psi_t^{\otimes N}$ for some time-dependent function $\psi_t \in L^2(\mathbb{R}^3)$ and show that formally, in the limit $N \to \infty$, ψ_t satisfies precisely (5).

(ii) Write the finite BBGKY hierarchy (in differential form) for the marginals $\{\gamma^{(k)}_{N,t}\}_{k=1}^{N}$ associated with the N-body Schrödinger equation (3).

(iii) Write the infinite BBGKY hierarchy (in differential form) for the marginals $\{\gamma^{(k)}_{\infty,t}\}_{k=1}^{\infty}$ associated with the non-linear Schrödinger equation (5).

(iv) Show that a solution to the infinite BBGKY hierarchy is $\{|\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}|\}_{k=1}^{\infty}$ where φ_t is the solution to the non-linear Schrödinger equation (5) with initial data $\varphi_t\big|_{t=0} = \varphi$.

(v) Write the Duhamel (integral) formula for both the hierarchies found in (ii) and in (iii).

(vi) Iterate n times (n arbitrary positive integer) the Duhamel formulas for $\gamma^{(k)}_{N,t}$ and for $\gamma^{(k)}_{\infty,t}$. (*Hint:* in the case discussed in class, with two-body interaction, it was convenient to distinguish, in the Duhamel formula for $\gamma^{(k)}_{N,t}$, two operators A and B acting on density matrices; in this case it is natural to introduce *three* operators, say, A, B, C, with analogous meaning. Moreover, in view of the limit $N \to \infty$ consider the terms of order $\frac{1}{N}$ and $\frac{k}{N}$ as small perturbations and stop the iteration of the Duhamel formula whenever you encounter a perturbation, as done in class in the case with two-body interaction.)

(vii) Given the expansions found in (vi), estimate the difference $\mathrm{Tr}|\gamma^{(k)}_{N,t} - \gamma^{(k)}_{\infty,t}|$ and show that there exists $t_0 > 0$, sufficiently small in units $\|V\|_\infty^{-1}$, such that if $t \in [0, t_0]$ then the limit (4) holds true.

(viii) Mimicking the counting discussed in the case of two-body interaction, show that the limit (4) holds true for any fixed time t and that the infinite BBGKY hierarchy has a unique solution, which in view of (iv) is therefore $\{|\varphi_t^{\otimes k}\rangle\langle\varphi_t^{\otimes k}|\}_{k=1}^{\infty}$. This concludes the proof of the Theorem.

(ix) (★) *OPTIONAL* Replace (2) with a more general potential of the form

$$W(x,y,z) \;=\; U(x-y,x-z) \;+\; U(x-y,y-z) \;+\; U(x-z,y-z) \qquad \forall\, x,y,z \in \mathbb{R}^3 \quad (6)$$

for some bounded, compactly supported $U : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ with $U(x,y) = U(-x,y) = U(x,-y)$ $\forall x,y \in \mathbb{R}^3$. Guess the form of the non-linear Schrödinger equation emerging in the construction above, that replaces (5).