Functional Analysis – Problems in the class, sheet 11

Mathematisches Institut der LMU – SS2010
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The problems in the class are discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for your preparation towards the final exam. Further info at `www.math.lmu.de/~michel/SS10_FA.html`.

NOTICE: last problem sheet!

Problem 41. Let $x = (x_1, x_2, x_3, \dots)$ be a sequence of complex numbers such that

$$\sum_{n=1}^{\infty} x_n y_n < \infty \qquad \text{for all sequences } y = (y_1, y_2, y_3, \dots) \text{ in } \ell^3.$$

Prove that x is in $\ell^{3/2}$.

Problem 42. Consider the linear map $(C^1([0,1]), \| \, \|_\infty) \xrightarrow{D} (C([0,1]), \| \, \|_\infty)$ given by $Df = f'$.

(i) Show that D is not continuous.

(ii) Show that $\Gamma(D)$, the graph of D, is closed.

(iii) Why do not (i)-(ii) contradict the closed graph theorem?

Problem 43. (*The goal of this problem is to show that there exists a continuous function f on $[-\pi, \pi]$ such that the partial sums f_N of its Fourier series,*

$$f_N(x) \equiv S_N(f)(x) = \frac{1}{2\pi} \sum_{n=-N}^{N} e^{-inx} \int_{-\pi}^{\pi} e^{iny} f(y) dy,$$

diverge at the point $x = 0$ in the limit $N \to \infty$.)

(i) Prove that if a linear functional l on $C[-\pi, \pi]$ is given by $l(f) = \int_{-\pi}^{\pi} \phi(x) f(x) \mathrm{d}x$ with some $\phi \in C[-\pi, \pi]$, then $\|l\| = \int_{\pi}^{\pi} |\phi(x)| \mathrm{d}x$.

(ii) Recall that partial sums of the Fourier series are given by convolution with the Dirichlet kernel, $f_N = f * D_N$, so that $f_N(0) = \int_{-\pi}^{\pi} D_N(x) f(x) \mathrm{d}x$. Use the explicit form of the Dirichlet kernel to show that if the linear functional l_N is defined by $l_N(f) = f_N(0)$, then $\|l_N\| \to +\infty$ as $N \to \infty$.

(iii) Use the Banach-Steinhaus theorem to conclude that there exist a continuous f such that the sequence $f_N(0), N = 1, 2, \dots$, is unbounded.

Problem 44. Let $(X, \| \, \|_X)$ be a normed vector space.

(i) Let $x \in X$ such that $|\phi(x)| \leqslant 1 \; \forall \phi \in X^*$ with $\|\phi\|_{X^*} \leqslant 1$. Prove that $\|x\|_X \leqslant 1$.

(ii) Prove that, for every $x \in X$, $\|x\|_X = \sup\limits_{\substack{\phi \in X^* \\ \|\phi\|_{X^*} = 1}} |\phi(x)|$.