Functional Analysis – Problems in the class, sheet 9

Mathematisches Institut der LMU – SS2010
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The problems in the class are discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for your preparation towards the final exam. Further info at `www.math.lmu.de/~michel/SS10_FA.html`.

Problem 33.

(i) Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Given $y = (y_1, y_2, y_3, \dots) \in \ell^q$ consider the map $\phi_y : \ell^p \to \mathbb{C}$ such that $\phi_y(x) := \sum_n x_n y_n$ for all $x = (x_1, x_2, x_3, \dots) \in \ell^p$. Prove that ϕ_y is a bounded linear functional on ℓ^p and that the correspondence $y \mapsto \phi_y$ is an isometric isomorphism $\ell^q \xrightarrow{\cong} (\ell^p)^*$.

(ii) Given $y = (y_1, y_2, y_3, \dots) \in \ell^1$ consider the map $\phi_y : \ell^\infty \to \mathbb{C}$ such that $\phi_y(x) := \sum_n x_n y_n$ for all $x = (x_1, x_2, x_3, \dots) \in \ell^\infty$. Prove that ϕ_y is a bounded linear functional on ℓ^∞, but the correspondence $y \mapsto \phi_y$ is only an isometry of ℓ^1 *into* $(\ell^\infty)^*$ ($y \mapsto \phi_y$ is not surjective), and therefore $\ell^1 \subsetneq (\ell^\infty)^*$ (in the sense that ℓ^1 is isometrically isomorphic to a proper subspace of $(\ell^\infty)^*$).

Problem 34.

(i) Give two examples of normed vector spaces X and Y such that $X \subset Y$ but $X^* \not\supset Y^*$ (therefore, the implication $X \subset Y \Rightarrow X^* \supset Y^*$ is tempting but false in general).

(ii) Let Y be a normed vector spaces and X a subspace of Y *dense* in Y. Show that $X^* \supset Y^*$. Give an example of such a situation.

Problem 35. Let $(X, \| \, \|_X)$ be a normed space and let $(X^*, \| \, \|_{X^*})$ be its dual. Let $\phi \in X^*$, $\phi \neq 0$. Prove that

$$\|\phi\|_{X^*} = \frac{1}{\inf\limits_{\substack{x \in X \\ \phi(x)=1}} \|x\|_X}$$

(i.e., the norm of ϕ is the reciprocal of the distance from zero to the hyperplane $\{\phi(x)=1\}$.)

Problem 36. Consider a collection $\{x_\alpha\}_{\alpha \in A}$ of vectors in some Hilbert space $\mathcal{H}$ and let $x \in \mathcal{H}$.

(i) Show that if $x_\alpha \to x$ weakly and $\|x_\alpha\| \to \|x\|$ then $x_\alpha \to x$ strongly.

(ii) Show that $x_\alpha \to x$ strongly if and only if $\langle y, x_\alpha \rangle \to \langle y, x \rangle$ uniformly for $\|y\| = 1$ (i.e., strong convergence is the same as weak convergence uniformly on the unit sphere).