Functional Analysis – Problems in the class, sheet 8

Mathematisches Institut der LMU – SS2010
Prof. L. Erdős Ph.D., A. Michelangeli Ph.D.

The problems in the class are discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for your preparation towards the final exam. Further info at `www.math.lmu.de/~michel/SS10_FA.html`.

Problem 29. (The heat equation on S^1.) Let $f_0 \in L^2(S^1)$, $f_0 = \sum_n c_n e_n$.

(i) Prove that $f(t,x) := \sum_n e^{-tn^2} c_n e_n(x)$ defines for every $t > 0$ a function $f(t,\cdot) \in C^\infty(S^1)$.

(ii) Prove that for every $x \in S^1$ the function $f(\cdot, x) \in C^1(\mathbb{R}^+)$.

(iii) Prove that f solves $\partial_t f = \Delta_x f$ for any $t > 0$ as an identity in $C^1(\mathbb{R}^+_t, C^2(S^1_x))$ – in fact, in $C^\infty(\mathbb{R}^+_t, C^\infty(S^1_x))$ – and satisfies the initial condition $\lim_{t\to 0^+} f(t,\cdot) = f_0$ where the limit is in $L^2(S^1)$.

Problem 30. Find the general solution in $C^2(S^1 \times S^1)$ to the partial differential equation

$$2f_{xx} + f_{xy} + f_{yy} \; = \; \cos x \cos y \, .$$

Hint: use the Fourier series of f.

Problem 31. Find the Fourier transform in $L^2(\mathbb{R})$ of the following functions:

(i) $f_1(x) = e^{-a|x|} \qquad (a > 0)$

(ii) $f_2(x) = \dfrac{1}{x^2 + a^2} \qquad (a > 0)$

(iii) $f_3(x) = \mathbb{1}_{[a,b]}(x) \qquad (a, b \in \mathbb{R},\ a < b)$

(iv) $f_4(x) = \dfrac{\sin ax}{x} \qquad (a > 0)\, .$

Problem 32. (Functions of rapid decrease.) Denote by $\alpha = \langle \alpha_1, \dots, \alpha_n \rangle$ an n-tuple of non-negative integers (n is a positive integer). Such α is said a n-dimensional *multi-index*. Its *size* is the number $|\alpha| := \sum_{j=1}^{n} \alpha_j$. Given $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and given a n-dimensional multi-index α, x^α will denote the product $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and D^α will denote the partial derivative $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. Define the space $\mathcal{S}(\mathbb{R}^n)$ of *functions of rapid decrease* as the space of smooth functions $f \in C^\infty(\mathbb{R}^n)$ such that

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| < \infty \qquad \text{for all multi-indices } \alpha \text{ and } \beta\,.$$

(i) Prove that $\| \ \|_{\alpha,\beta}$ satisfies

- $\|f + g\|_{\alpha,\beta} \leqslant \|f\|_{\alpha,\beta} + \|g\|_{\alpha,\beta}$
- $\|\lambda f\|_{\alpha,\beta} = |\lambda| \|f\|_{\alpha,\beta}$

for all $f, g \in \mathcal{S}(\mathbb{R}^n)$, all $\lambda \in \mathbb{C}$, and all multi-indices α, β, and moreover

- $(\|f\|_{\alpha,\beta} = 0 \ \forall \alpha, \beta) \Rightarrow (f(x) = 0 \ \forall x \in \mathbb{R}^n)$.

(ii) Equip $\mathcal{S}(\mathbb{R}^n)$ with the topology generated by the sub-basis of neighbourhoods at 0 consisting of the neighbourhoods of the form

$$\mathcal{N}_{\alpha,\beta,\varepsilon} := \left\{ f \in \mathcal{S}(\mathbb{R}^n) \mid \|f\|_{\alpha,\beta} < \varepsilon \right\}.$$

Prove that

- $\| \ \|_{\alpha,\beta}$ is continuous for all multi-indices α, β
- the pointwise sum of two functions of rapid decrease is a continuous map $\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$
- $f_n \xrightarrow{n \to \infty} f$ in the topology of $\mathcal{S}(\mathbb{R}^n)$ if and only if $\|f_n - f\|_{\alpha,\beta} \xrightarrow{n \to \infty} 0$ for all multi-indices α, β.

(iii) Prove that for any $p \in [1, \infty]$ the identity map $id : \mathcal{S}(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is continuous.

(iv) Prove that the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ induces a bijection of $\mathcal{S}(\mathbb{R}^n)$ onto itself.