Functional Analysis – Problems in the class, sheet 6

Mathematisches Institut der LMU – SS2010
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The problems in the class are discussed interactively in the weekly exercise/tutorial sessions. You are not required to hand in their solution. You are encouraged to think them over and to solve them. Being able to solve them is essential for your preparation towards the final exam. Further info at `www.math.lmu.de/~michel/SS10_FA.html`.

Problem 21.

(i) Consider the system $\{\varphi_{mn} \mid m \in \mathbb{N}, 1 \leqslant n \leqslant 2^m\}$ of $[0,1] \to \mathbb{R}$ functions defined by

$$\varphi_{mn}(x) \ := \begin{cases} 2^{m/2} & \text{if } \dfrac{n-1}{2^m} \leqslant x \leqslant \dfrac{n-\frac{1}{2}}{2^m} \\ -2^{m/2} & \text{if } \dfrac{n-\frac{1}{2}}{2^m} \leqslant x \leqslant \dfrac{n}{2^m} \\ 0 & \text{otherwise}\,. \end{cases}$$

Prove that this system is an orthonormal basis in $L^2[0,1]$ (the HAAR BASIS).

(ii) Show that the RADEMACHER SYSTEM $\{\varphi_n\}_{n\in\mathbb{N}}$, $\varphi_n(x) := (-1)^{[2^m x]}$, is orthonormal but not dense in $L^2[0,1]$ (here $[z]$ denotes the integral part of the real number z).

(iii) Show that the system

$$\Big\{\ \varphi_{m_1\cdots m_n} = \prod_{k=1}^{n} \varphi_{m_k} \ \Big|\ m_1 < \cdots < m_n\,,\ \varphi_{m_k} \text{ is a Rademacher function}\ \Big\}$$

is an orthonormal basis in $L^2[0,1]$ (the WALSH BASIS).

Problem 22. Compute the angles of the triangle formed by the following points in $L^2[-1,1]$:

$$f_1(x) = 0\,, \qquad f_2(x) = 1\,, \qquad f_3(x) = x\,.$$

Problem 23.

(i) Let $g \in L^2[0,2\pi]$. Show that $\displaystyle\int_0^{2\pi} g(x)\, e^{-inx}\, \mathrm{d}x \to 0$ as $n \to \infty$

(ii) Show that the same holds when $g \in L^1[0,2\pi]$ too.

Problem 24. Compute the distance in $L^2[0,1]$ of the exponential function $x \mapsto e^x$ from the subspace of polynomials of degree $\leqslant 3$. Is the minimum attained?